Pricing of Variance Swaps under a Credit-Equity Modeling Framework

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Abstract

We compute the value of a variance swap when the underlying is modeled as a Markov process time changed by a Lévy subordinator. In this framework, the underlying may exhibit jumps with a state-dependent Lévy measure, local stochastic volatility and have a local stochastic default intensity. Moreover, the Lévy subordinator that drives the underlying can be obtained directly by observing European call/put prices. To illustrate our general framework, we provide an explicit formula for the value of a variance swap when the diffusion is modeled as (i) a Lévy subordinated geometric Brownian motion with default and (ii) a Lévy subordinated Jump-to-default CEV process (see Carr and Linetsky (2006)). Our results extend the results of Mendoza-Arriaga et al. (2010), by allowing for joint valuation of credit and equity derivatives as well as variance swaps.

1 Introduction

A *variance swap* (VS) is a forward contract written on the realized variance of an underlying $S = \{S_t, t \geq 0\}$. As is typical in derivatives literature, we define the *realized variance* over the interval $[0, t]$ as $[\log S]_t$, the continuously sampled quadratic variation of the $\log S$. Thus, at maturity the VS has a payoff (to the long side) of

$$[\log S]_t - K_{var}. \tag{1.1}$$

The *variance swap rate* $K_{var} = \mathbb{E}\left([\log S]_t\right)$ is determined at inception so that initial value of the VS is zero. Here, $\mathbb{E}$ is the expectation under the risk-neutral pricing measure $\mathbb{P}$.

There are a number of reasons for which one may wish to enter into a VS agreement. First, a trader who delta-hedges a short position in a European option can limit his exposure to stochastic volatility risk by trading a VS (see. e.g. Carr and Schoutens (2008)). Second, a drop in the level of an underlying S is often accompanied by an increase in the volatility of S (the leverage effect). Thus, a long position in a VS serves as protection against a market crash. Such is the demand for VSs that, according to Jung (2006), the daily trading volume in equity index VSs reached 45 million USD vega notional in 2006 (vega measures the change in an option's price caused by changes in volatility). On an annual basis, this corresponds to payments of more than 1 billion USD per percentage point of volatility (see Carr and Lee (2009)).

At the individual stock level there is yet another source for the leverage effect that is related to the default risk associated with the underlying firm. This credit-related leverage effect explains the interaction between market risk (return variance) and credit risk (default arrival). Carr and Wu (2009) recently studied this interaction in pricing stock options and credit default swaps. In general, when the probability of default of a firm increases, its stock price tends to lose value while the option-implied volatility increases. In such a

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situation, it is common for investors to look for credit protection by taking long positions in deep out-of-the-money puts. However, there is a side-effect in this hedging strategy. Though the investor's objective is to minimize credit exposure, this strategy will increase his exposure to volatility due to the credit-equity leverage effect. Hence, in the pricing of variance swaps written on an individual stock it is especially important to take into account default risk (and the credit-related leverage effect).

In his seminal paper, Neuberger (1990) showed that, when the underlying S is modeled by a process with *continuous* sample paths, $dS_t = \sigma_t S_t dW_t$, (for simplicity, assume the risk-free rate of interest is zero) the fair value of K_{var} is given by a European-style log contract $K_{var} = \mathbb{E}[-2\log(S_t/S_0)]$. Later, Carr and Madan (1998) showed that the log contract (or in fact, any European-style derivative with a twice differentiable payoff) can be synthesized from a linear combination of calls and puts. [1] Thus, when t-expiry calls and/or puts are available at every $K \in (0, \infty)$, the fair value of K_{var} is uniquely determined by the implied volatility smile. Remarkably, this result is independent of any assumption about the volatility process $\sigma = \{\sigma_t, t \geq 0\}$. However, this result does rely on the continuity of sample paths of S. The events of the recent (and ongoing) financial crisis underscore the need to include jumps in the underlying price process S. The question naturally arises then: what is the fair value of K_{var} when the price process S is allowed to have discontinuous sample paths?

One possible answer to this question is given by Carr et al. (2012), who show that, when $\log S$ is modeled as a Lévy process time-changed by an absolutely continuous time-change, the fair value of K_{var} is equal to a multiplier Q_X times the value of a log contract $K_{var} = Q_X \mathbb{E}[-\log(S_t/S_0)]$. Interestingly, the multiplier Q_X does *not* depend on the time-change process. While the value of K_{var} *does* depend on the time-change through the log contract, since the log contract can be synthesized by a linear combination of calls, when one has full knowledge of the volatility smile, the value of K_{var} can be determined in the framework of Carr et al. (2012) without any knowledge of the time-change process.

This last point cannot be emphasized enough. A parametric model for the time-change process would leave one open to model risk, as any misspecification of the time-change parameters would (in general) result in erroneous values of K_{var}. By using knowledge of the volatility smile to construct the value of the log contract, Carr et al. (2012) circumvent the need to parametrically model the time-change process. Thus, the risk of model misspecification is greatly reduced (though, some model risk still exists, since the multiplier Q_X depends on the choice of a specific Lévy process). An alternative and quite interesting approach to the joint pricing of volatility derivatives and index options in the presence of jumps is provided in Cont and Kokholm (2011).

In this paper, we consider the class of Lévy subordinated diffusion processes described in Mendoza-Arriaga et al. (2010). This class of models, like the class considered by Carr et al. (2012), allows for the underlying S to experience both jumps and stochastic volatility. However, there are a few important differences between these two frameworks. In Carr et al. (2012), the background process is modeled as a Lévy process, which naturally includes the possibility of jumps, but does not include stochastic volatility. Stochastic volatility is added by time-changing the background process with a continuous increasing stochastic clock. In contrast, in Mendoza-Arriaga et al. (2010), the background process is modeled as a diffusion, which may include local stochastic volatility, but does not include jumps. Jumps are added by time-changing the background process with a Lévy subordinator. Additionally, the framework of Mendoza-Arriaga et al. (2010) allows for the possibility of a default event, whereas the framework of Carr et al. (2012) does not. While default may not be realistic for an index, it is certainly an important consideration for individual stocks as explained above.[2]

Despite the differences between these frameworks, they share one desirable feature: when the background process is fixed and one has full knowledge of the volatility smile, the fair value of K_{var} is robust to misspecification of the time-change process. In Carr et al. (2012), the effect that the time-change has on the value of K_{var} is felt through the log contract, which is constructed directly from European calls. In the framework of Mendoza-Arriaga et al. (2010), we will show that the Lévy subordinator can be inferred directly from the volatility smile. Once the subordinator is obtained, it can be used to compute the fair value of K_{var} (among

[1] In fact, the replication result of Carr and Madan (1998) is independent of the continuity assumption of the price process.

[2] A default event would cause $[\log S]_t$ to blow up. As such, we must amend our definition of a VS to account for this possibility. We will do this in section 4.

other things).

The rest of this paper proceeds as follows. In section 2 we describe the class of Lévy subordinated diffusion processes in detail. In section 3 we introduce some mathematical tools, which we shall need to compute the price of a VS. In section 4 we modify the payoff of a VS to account for the possibility that the underlying defaults (i.e., jumps to zero). We then derive a general expression for the value of the modified VS in the Lévy subordinated diffusion setting. In section 5 we present some important results concerning generalized eigenfunction expansions. These results will be needed for the diffusion-specific VS computations provided in section 6. In section 7 we show that, by observing European call and put prices written on the underlying, one can uniquely determine the drift and Lévy measure of the subordinator that drives the underlying price process. Finally, in section 8, we provide some concluding remarks and discuss directions for future research.

2 Model

We assume frictionless markets, no arbitrage, and take an equivalent martingale measure (EMM) $\mathbb{P}$ chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$ as given. All stochastic processes defined below live on this probability space, and all expectations are with respect to $\mathbb{P}$ unless stated otherwise. We also assume that all the (natural) filtrations defined throughout this section are augmented to include the $\mathbb{P}$-null sets. In particular, this assumption holds for the *subordinate filtration* $\mathcal{F}_t$, which we describe at the end of this section. For simplicity, we assume zero interest rates. All of our results can be easily extended to include deterministic interest rates.

As in Mendoza-Arriaga et al. (2010), we model the stock price dynamics under the risk-neutral pricing measure $\mathbb{P}$ as a stochastic process $S = \{S_t, t \geq 0\}$ defined by

$$S_t = (1 - D_{T_t}) e^{\rho t} X_{T_t}, \qquad D_t = \mathbb{I}_{\{\zeta \leq t\}}, \qquad X_0 = x, \qquad T_t = 0, \qquad \zeta > 0. \tag{2.1}$$

Here, the background process $X = \{X_t, t \geq 0\}$ is a scalar Feller diffusion, $T = \{T_t, t \geq 0\}$ is a Lévy subordinator, ρ is a scaling factor, which is needed to ensure that the asset price S is a martingale, and ζ is a positive random variable, which will be used to model a possible default event of the underlying S. Below, we describe each of the above-mentioned elements in detail. Additionally, we discuss how some important filtrations relate to a trader's observation of the market.

Background Feller process X**.** We let $X = \{X_t, t \geq 0\}$ be a time-homogeneous Markov diffusion process, starting from a positive value $X_0 = x > 0$, which solves a stochastic differential equation (SDE) of the form

$$dX_t = b(X_t)\, dt + a(X_t)\, dB_t, \tag{2.2}$$

where

$$b(x) := [\mu + k(x)]x \qquad \text{and} \qquad a(x) := \sigma(x)x. \tag{2.3}$$

Here, $\sigma(x)$ and $[\mu + k(x)]$ are the state-dependent instantaneous volatility and drift rate, $\mu \in \mathbb{R}$ is a constant, and $B = \{B_t, t \geq 0\}$ is a standard Brownian motion. We assume that $\sigma(x) > 0$ and $k(x) > 0$ are Lipschitz continuous on the interval $[\varepsilon, \infty)$ for each $\varepsilon > 0$ (i.e., locally Lipschitz), and that $\sigma(x)$ and $k(x)$ remain bounded as $x \to \infty$. We do not assume that $\sigma(x)$ and $k(x)$ remain bounded as $x \to 0$. Under these assumptions the process X does not explode to infinity (i.e., infinity is a *natural boundary* for the diffusion process; see Borodin and Salminen (2002) p.14 for boundary classification of diffusion processes). We also assume that zero is either an (unattainable) natural boundary or an entrance boundary. If zero is a natural boundary the state space is given by $E = (0, \infty)$. If zero is an entrance boundary, i.e., the process X can be started at $x = 0$ then it quickly moves to interior of $[0, \infty)$ to never hit zero again. Throughout this document we assume that the process always starts from a positive value $X_0 = x > 0$, and hence the state space is also defined as $E = (0, \infty)$. Under all our previous assumptions X is the unique strong solution to the SDE (2.2). The transition function $P_t^0(x, dy) = \mathbb{P}(X_t \in dy | X_0 = x)$ of the diffusion process X started

at x defines a Feller semigroup $\mathcal{P}^0 = \{\mathcal{P}^0_t, t \geq 0\}$ acting on the space $C([0,\infty])$ of functions continuous on $(0,\infty)$ and such that the limits $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$ exist and are finite (Ethier and Kurtz (1986) p.366) by

$$\mathcal{P}^0_t f(x) = \mathbb{E}_x\big[f(X_t)\big] = \int_E f(y) P^0_t(x,dy). \tag{2.4}$$

The infinitesimal generator of $\mathcal{P}^0$ is a second-order differential operator of the form

$$\mathcal{A}^0 f(x) = \frac{1}{2}a^2(x)\frac{d^2 f}{dx^2}(x) + b(x)\frac{df}{dx}(x)$$

with the domain $\mathrm{Dom}(\mathcal{A}^0) = \{f \in C([0,\infty]) \cap C^2((0,\infty)), \mathcal{A}^0 f \in C([0,\infty])\}$ if zero is an inaccessible boundary (natural or entrance). We also note that the semigroup leaves the space $C_0((0,\infty)) \subset C([0,\infty])$ of functions continuous on $(0,\infty)$ and having zero limits $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$ invariant and is a Feller semigroup on it. Lastly, we denote by $\mathcal{F}^X$ the natural filtration of the process X.

The trigger event ζ and the indicator process D. Let $\mathcal{E} \sim \mathrm{Exp}(1)$ be an exponential random variable, independent of X, and define the *trigger event time* ζ as

$$\zeta = \inf\Big\{t \geq 0 : \int_0^t k(X_s)ds \geq \mathcal{E}\Big\}.$$

That is, ζ is the first jump time of a doubly stochastic Poisson process with jump intensity given by the killing rate $k(x)$. Observe that the killing rate is added in the drift (2.3) to compensate for the killing (jump-to-default). This compensation will be needed in order to ensure that the stock price S is a martingale. Moreover, since the process X cannot reach zero from the interior of the state space, then the underlying stock price process S cannot go to zero continuously, rather, it may only jump to zero from a strictly positive value. We denote by $\mathcal{P}^1 = \{\mathcal{P}^1_t, t \geq 0\}$ the Feynman-Kac semigroup associated with the killing rate $k(X)$:

$$\mathcal{P}^1_t f(x) = \mathbb{E}_x\big[e^{-\int_0^t k(X_u)du} f(X_t)\big] = \int_E f(y) P^1_t(x,dy), \quad t \geq 0. \tag{2.5}$$

$\mathcal{P}^1$ is a (sub-Markovian) Feller semigroup on $C([0,\infty])$ with the generator

$$\mathcal{A}^1 f(x) = \mathcal{A}^0 f(x) - k(x) f(x) \tag{2.6}$$

with the domain $\mathrm{Dom}(\mathcal{A}^1) \subseteq \mathrm{Dom}(\mathcal{A}^0)$. More precisely, $\mathrm{Dom}(\mathcal{A}^1) = \{f \in C([0,\infty]) \cap C^2((0,\infty)), \mathcal{A}^1 f \in C([0,\infty])\}$. Since zero is an inaccessible boundary (natural or entrance) of the *diffusion with killing at the rate* $k(x)$ it suffices to restrict the domain to $\mathrm{Dom}(\mathcal{A}^1) = \{f \in C_0([0,\infty]) \cap C^2((0,\infty)), \mathcal{A}^1 f \in C_0([0,\infty])\}$ whenever we start the process from the interior of E, i.e., $x > 0$ (cf. Borodin and Salminen (2002) pp.15ff).

Since the random variable ζ is not $\mathcal{F}^X$ measurable, we introduce an *indicator process* $D = \{D_t, t \geq 0\}$ in order to keep track of the event $\{\zeta \leq t\}$. The indicator process D is defined by

$$D_t = \mathbb{I}_{\{\zeta \leq t\}}, \qquad (1 - D_t) = \mathbb{I}_{\{\zeta > t\}}. \tag{2.7}$$

Lastly, we denote by $\mathcal{F}^D$ to the natural filtration of the process D.

The auxiliary bivariate process (X,D). As in Mendoza-Arriaga and Linetsky (2012) we define the auxiliary bivariate process $(X,D) = \{(X_t, D_t), t \geq 0\}$ with state space $\tilde{E} = E \times \{0,1\}$. The process (X,D) is a Feller semimartingale in the enlarged filtration $\mathcal{F}^{X,D} = \mathcal{F}^X \vee \mathcal{F}^D$ ($\mathcal{F}^{X,D}$ is the smallest filtration that contains $\mathcal{F}^X$ and in which ζ is a stopping time). Moreover, any function $f(x,d) \in C([0,\infty] \times \{0,1\})$ is represented as,

$$f(x,d) = u(x)(1-d) + v(x)d = (u-v)(x)(1-d) + v(x), \qquad u, v \in C([0,\infty]). \tag{2.8}$$

As we shall see below, the function $u(x) = f(x,0)$ can be interpreted as a promised payoff function if the triggering event ζ *does not* occur by time $t > 0$, while $v(x) = f(x,1)$ can be understood as a recovery payoff function if the triggering event occurs prior time $t > 0$.

The following theorem gives the Markovian characterization of the bivariate process (X,D).

Theorem 2.1. *(i) The bi-variate process (X, D) is a Feller process with the Feller semigroup $\mathcal{P} = \{\mathcal{P}_t, t \geq 0\}$ acting on $f \in C([0,\infty] \times \{0,1\})$ according to:*

$$\mathcal{P}_t f(x,d) = \mathcal{P}_t^0 v(x) + (1-d)\mathcal{P}_t^1 (u-v)(x),$$

where $u, v \in C([0,\infty])$, $\mathcal{P}^0$ is the Feller semigroup of the process X (2.4) and $\mathcal{P}^1$ is the corresponding Feynman-Kac semigroup (2.5).

(ii) The infinitesimal generator of the Feller semigroup $\mathcal{P}$ is given by

$$\mathcal{A}f(x,d) = \mathcal{A}^0 f(x,d) + k(x)(f(x,1) - f(x,d))$$

$$= \mathcal{A}^0 v(x) + (1-d)\mathcal{A}^1 (u-v)(x),$$

where $\mathcal{A}^0$ and $\mathcal{A}^1$ are the generators of $\mathcal{P}^0$ and $\mathcal{P}^1$, respectively.

(iii) If $f(x,d) \in \mathrm{Dom}(\mathcal{A})$ (i.e., f is of the form (2.8) with $u, v \in \mathrm{Dom}(\mathcal{A}^1)$) and (X, D) is the bi-variate process with $X_0 = x > 0$ and $D_0 = d \in \{0,1\}$, then the process

$$M_t^f := f(X_t, D_t) - f(x,d) - \int_0^t \mathcal{A}f(X_s, D_s)ds$$

is a $\mathcal{F}^{X,D}$-martingale.

Proof. The proof can be found in Mendoza-Arriaga and Linetsky (2012). □

Similarly, the Doob-Meyer decomposition of D is given in the following corollary.

Corollary 2.2. *The increasing process D has the Doob-Meyer decomposition*

$$D_t = M_t + A_t$$

with the predictable $\mathcal{F}^{X,D}$-compensator A_t and $\mathcal{F}^{X,D}$-martingale M_t

$$A_t = \int_0^t k(X_u)(1-D_u)du, \quad M_t = D_t - \int_0^t k(X_u)(1-D_u)du.$$

Proof. See section 5.1.4 in Bielecki and Rutkowski (2004) (also, Lemma 7.3.4.3 and Comment 7.3.4.4 in Jeanblanc et al. (2009)). □

The Lévy subordinator T. A *Lévy subordinator* $T = \{T_t, t \geq 0\}$ is a Lévy process with positive jumps and non-negative drift. For a standard reference on subordinators we refer the reader to Bertoin (2004). We require that T be independent of X and $\mathcal{E}$ and satisfy $T_0 = 0$. Every Lévy subordinator has the following *Itô-Lévy decomposition*

$$dT_t = \gamma\, dt + \int_{(0,\infty)} s\, dN_t(ds),$$

where $\gamma \geq 0$ is the *drift* of the subordinator and N is a *Poisson random measure* with the property that, for any Borel set $A \in \mathcal{B}((0,\infty))$ we have $\mathbb{E}[dN_t(A)] = \nu(A)dt$ for some σ-finite measure ν on $\mathcal{B}((0,\infty))$. The measure ν, which must satisfy $\int_0^\infty (s \wedge 1)\, \nu(ds) < \infty$, is referred to as the *Lévy measure*. The Laplace transform of a Lévy subordinator T is given by

$$\mathbb{E}\left[e^{\lambda T_t}\right] = \int_0^\infty e^{\lambda s}\pi_t(ds) = e^{-t\phi(-\lambda)}, \qquad \lambda \in \left\{\lambda \in \mathbb{R} : \int_{[1,\infty)} e^{\lambda s}\nu(ds) < \infty\right\} =: \mathcal{I}, \tag{2.9}$$

where $\pi_t(ds)$ is the transition function of the subordinator T and $\phi(\lambda)$ is the *Laplace exponent* of T, which can be computed explicitly from the *Lévy-Kintchine formula*

$$\phi(\lambda) = \gamma\,\lambda + \int_{(0,\infty)} \left(1 - e^{-\lambda s}\right) \nu(ds). \tag{2.10}$$

Note that the Laplace exponent $\phi(\lambda)$ is concave and increasing and satisfies $\phi(0) = 0$ (see Bertoin (1998), page 73). For any Borel set $A \in \mathcal{B}((0,\infty))$ the process $\{N_t(A), t \geq 0\}$ is a Poisson process with intensity $\nu(A) = \int_A \nu(ds)$. If the arrival rate of all jumps is finite $\alpha := \nu((0,\infty)) < \infty$ then $\int_0^T \int_0^\infty s\, dN_t(ds)$ is a compound Poisson process. In this case, the distribution of jumps is $\alpha^{-1}\nu$. If $\nu((0,\infty)) = \infty$ then the subordinator is said be an *infinite activity* subordinator.

We define $L = \{L_t, t \geq 0\}$ the *first passage process* or *right inverse process* of T as $L_t := \inf\{s : T_s > t\}$. Recall that T is assumed to be *independent* of $\mathcal{F}^{X,D}_\infty$. Therefore, L is independent of $\mathcal{F}^{X,D}_\infty$ as well. We let $\mathcal{F}^L$ be the natural filtration of the process L_t.

The subordinate filtration $\mathcal{F}_t$, the auxiliary subordinate bivariate process (X^ϕ, D^ϕ), and the default time ζ^ϕ. Recall from our above discussion that $\mathcal{F}^X$, $\mathcal{F}^D$ and $\mathcal{F}^L$ correspond to the filtrations generated by the processes X, D, and L, respectively, and that the filtration $\mathcal{F}^{X,D} = \mathcal{F}^X \vee \mathcal{F}^D$ is the smallest filtration that contains $\mathcal{F}^X$ and in which ζ is an $\mathcal{F}^{X,D}$-stopping time. Similarly, we can define the filtration $\mathcal{F}^G = \mathcal{F}^{X,D} \vee \mathcal{F}^L$ to be the smallest filtration that contains $\mathcal{F}^{X,D}$ and in which T_t is an increasing family of $\mathcal{F}^G$-stopping times. Then the *subordinate filtration* $\mathcal{F}_t$ is constructed by time-changing the filtration $\mathcal{F}^G$ with the Lévy subordinator T, i.e., $\mathcal{F}_t = \mathcal{F}^G_{T_t}$. Observe that since T is an $\mathcal{F}^G$-stopping time, then $\mathcal{F}_t$ is the filtration containing all of the information of the bivariate process (X, D) prior to the stopping time T_t. Consequently, one can define the *subordinate bivariate process* $(X^\phi, D^\phi) := \{(X_{T_t}, D_{T_t}), t \geq 0\}$ by time-changing the bivariate process (X, D) with the subordinator T. This transformation is called *Bochner's subordination* due to work on subordination of semigroups and their generators by Bochner (1949). The subordinate process (X^ϕ, D^ϕ) is a Feller $\mathcal{F}$-semimartingale (see Mendoza-Arriaga and Linetsky (2012)).

From (2.7) we observe that, before subordination, the indicator process D satisfies $D_t = 1$ for all $t \geq \zeta$. On the other hand, after subordination, the subordinate indicator process D^ϕ satisfies $D^\phi_t = 1$ for all $t \geq 0$ for which $T_t \geq \zeta$ (i.e., $D^\phi_t = \mathbb{I}_{\{T_t \geq \zeta\}}$). Therefore, in the credit-equity context one can define the *default time* ζ^ϕ by

$$\zeta^\phi = L_{\zeta-}, \qquad\qquad L_{a-} = \inf\{t : T_t \geq a\}.$$

Certainly, ζ^ϕ is the first passage time process of T across the level ζ and the identity $\{\zeta^\phi \leq t\} = \{T_t \geq \zeta\}$ holds (see section 5.1.1 in Jeanblanc et al. (2009)). Hence, $D^\phi_t = \mathbb{I}_{\{T_t \geq \zeta\}} = \mathbb{I}_{\{\zeta^\phi \leq t\}}$ is the *default indicator process*. The characterization of the subordinate process (X^ϕ, D^ϕ) is provided below in section 3.

The stock price S and the scaling constant ρ. From Eq. (2.1) we observe that the dynamics of the stock price S can be described by means of the subordinate bivariate process (X^ϕ, D^ϕ). Indeed, the stock price S can be seen as a function $f(t, X^\phi_t, D^\phi_t) \in C([0,\infty) \times \tilde{E})$ which is decomposed according to (2.8) with the payoff function $u(t, x) = e^{\rho t} x$ if no default occurs by time $t \geq 0$, and zero recovery $v(t, x) = 0$ if the firm defaults prior to time $t \geq 0$. That is,

$$S_t = e^{\rho t} X^\phi_t (1 - D^\phi_t). \tag{2.11}$$

The scaling constant ρ is introduced to ensure that the asset S is an $\mathcal{F}$-martingale. As shown in Mendoza-Arriaga et al. (2010), S will be a martingale if and only if $\rho = \phi(-\mu)$ and $\mu \in \mathcal{I}$, where the set $\mathcal{I}$ is defined in Eq. (2.9). That is, assuming zero interest rates, $\mathbb{E}[S_t] < \infty$ for every $t \geq 0$, and $\mathbb{E}[S_{t_2} | \mathcal{F}_{t_1}] = S_{t_1}$ for every $t_1 < t_2$. From the previous condition in ρ we are free to choose any value of μ as long as $\mu \in \mathcal{I}$. Hence, from this point onward we assume that $\mu \in \mathcal{I}$. Observe that the underlying assumption $v(x) = 0$ is equivalent to modeling the stock price S under *absolute priority*, which means that the stock holder has zero recovery in the event of default.

3 Markovian and Semimartingale Characterization of the Subordinate Process (X^ϕ, D^ϕ)

Before proceeding with the calculation of the quadratic variation of the log price process $\log S$ it is essential to describe the characteristics of the underlying stock process S in terms of the subordinate bivariate process (X^ϕ, D^ϕ). For convenience, we summarize some of the key results of section 3 in Mendoza-Arriaga and Linetsky (2012) who give the Markovian and semimartingale characterization of the process (X^ϕ, D^ϕ). We refer the reader to Mendoza-Arriaga and Linetsky (2012) for the corresponding proofs.

We begin by recalling some key results about subordination (in the sense of Bochner (1949)) of semigroups of operators in Banach spaces. The expression for the generator is due to Phillips (1952).

Theorem 3.1. (Phillips (1952)) *Let T be a subordinator with Lévy measure ν, drift $\gamma \geq 0$, Laplace exponent $\phi(\lambda)$, and transition function $\pi_t(ds)$. Let $\mathcal{P}$ be a strongly continuous contraction semigroup of linear operators on a Banach space $\mathfrak{B}$ with infinitesimal generator $\mathcal{A}$.*

(i) Define

$$\mathcal{P}^\phi_t f(x) = \int_{[0,\infty)} \mathcal{P}_s f(x) \pi_t(ds), \quad t \geq 0, \ f \in \mathfrak{B}. \tag{3.1}$$

Then $\mathcal{P}^\phi = \{\mathcal{P}^\phi_t, t \geq 0\}$ is a strongly continuous contraction semigroup of linear operators on $\mathfrak{B}$ called subordinate semigroup of $\mathcal{P}$ with respect to the subordinator T.

(ii) Denote the infinitesimal generator of $\mathcal{P}^\phi$ by $\mathcal{A}^\phi$. Then the domain of $\mathcal{A}$ is a core of $\mathcal{A}^\phi$ and

$$\mathcal{A}^\phi f = \gamma \mathcal{A} f + \int_{(0,\infty)} (\mathcal{P}_s f - f)\nu(ds), \quad f \in \mathrm{Dom}(\mathcal{A}).$$

(iii) Moreover, if $\mathcal{P}$ is a Feller semigroup on $C([0,\infty])$, then the subordinate semigroup $\mathcal{P}^\phi$ is also a Feller semigroup on $C([0,\infty])$.

Next, recall that if 0 is not an absorbing boundary for the process X with diffusion coefficient $a(x)$, drift $b(x)$ and killing rate $k(x)$, then the transition kernels $P^\beta(x,dy)$, $\beta = 0,1$, of the semigroups $\mathcal{P}^\beta$ have densities with respect to the Lebesgue measure, $P^\beta_t(x,dy) = p^\beta(t,x,y)dy$, where $p^\beta(t,x,y)$ are jointly continuous in t,x,y. This follows from the fact that any one-dimensional diffusion has a density with respect to the speed measure that is jointly continuous in t,x,y (cf. McKean (1956) or Borodin and Salminen (2002) p.13). Under our assumptions, the speed measure is absolutely continuous with respect to the Lebesgue measure (cf. Borodin and Salminen (2002), p.17) and, hence, the semigroups have densities with respect to the Lebesgue measure. For $\beta = 0$, the transition kernel is conservative, i.e., $P^0_t(x,E) = \int_E p^0(t,x,y)dy = 1$. For $\beta = 1$ the kernel is generally defective, i.e., $P^\beta_t(x,E) = \int_E p^1(t,x,y)dy \leq 1$. While our diffusion is non-negative, for future convenience we extend the transition densities from E to $\mathbb{R}$ by setting $p^\beta(t,x,y) \equiv 0$ for all $y \notin E$ and for all $x \in E$ and $t \geq 0$. Then, the Markovian characterization of the subordinate bivariate process (X^ϕ, D^ϕ) can be obtained from Theorem 3.1 as follows.

Theorem 3.2. (Markovian characterization of (X^ϕ, D^ϕ)) *(i) The bi-variate process (X^ϕ, D^ϕ) is a Feller process with the Feller semigroup $\{\mathcal{P}^\phi_t, t \geq 0\}$ acting on $f \in C([0,\infty] \times \{0,1\})$ by:*

$$\mathcal{P}^\phi_t f(x,d) = \mathcal{P}^{\phi,0}_t v(x) + (1-d)\mathcal{P}^{\phi,1}_t (u-v)(x),$$

where $u(x) = f(x,0) \in C([0,\infty])$, $v(x) = f(x,1) \in C([0,\infty])$, and $\{\mathcal{P}^{\phi,0}_t, t \geq 0\}$ and $\{\mathcal{P}^{\phi,1}_t, t \geq 0\}$ are Feller semigroups obtained by subordination in the sense of Bochner from Feller semigroups $\{\mathcal{P}^0_t, t \geq 0\}$ and $\{\mathcal{P}^1_t, t \geq 0\}$.

(ii) The infinitesimal generator $\mathcal{A}^\phi$ of the Feller semigroup $\{\mathcal{P}^\phi_t, t \geq 0\}$ has the following representation:

$$\mathcal{A}^\phi f(x,d) = \mathcal{A}^{\phi,0} v(x) + (1-d)\mathcal{A}^{\phi,1}(u-v)(x), \quad u,v \in \mathrm{Dom}(\mathcal{A}^1), \tag{3.2}$$

where $\mathcal{A}^{\phi,\beta}$, $\beta \in \{0,1\}$, are generators of $\{\mathcal{P}_t^{\phi,\beta}, t \geq 0\}$.

(iii) The generator $\mathcal{A}^{\phi,\beta}$ has the following Lévy-Khintchine-type representations with state-dependent coefficients:

$$\mathcal{A}^{\phi,\beta} f(x) = \frac{1}{2}\gamma a^2(x) f''(x) + b^{\phi,\beta}(x) f'(x) - \beta k^{\phi}(x) f(x) \\ + \int_{\mathbb{R}} \Big(f(x+y) - f(x) - \mathbb{I}_{\{|y|\leq 1\}} y f'(x) \Big) \pi^{\phi,\beta}(x,y) dy, \quad f \in \mathrm{Dom}(\mathcal{A}^{\beta}),$$

where the state-dependent Lévy density $\pi^{\beta}(x,y)$ is defined for all $y \neq x$ by

$$\pi^{\phi,\beta}(x,y) = \int_{(0,\infty)} p^{\beta}(s,x,x+y)\nu(ds), \tag{3.3}$$

and satisfies the integrability condition $\int_{\mathbb{R}}(|y|^2 \wedge 1)\pi^{\phi,\beta}(x,y)dy < \infty$ for each $x \in E$ (recall that we extended $p^{\beta}(t,x,y)$ to $\mathbb{R}$ by setting $p^{\beta}(t,x,y) \equiv 0$ for $y \notin E$), the drift with respect to the truncation function $h^{X^{\phi}}(x) = x\mathbb{I}_{\{|x|\leq 1\}}$ is given by,

$$b^{\phi,\beta}(x) = \gamma b(x) + \int_{(0,\infty)} \Big(\int_{\{|y|\leq 1\}} y p^{\beta}(s,x,x+y) dy \Big) \nu(ds), \tag{3.4}$$

and the killing rate is given by

$$k^{\phi}(x) = \gamma k(x) + \int_{(0,\infty)} \Big(1 - P_s^{\beta}(x,E) \Big) \nu(ds), \tag{3.5}$$

where $P_s^{\beta}(x,E) = \int_E p^{\beta}(s,x,y)dy$.

(iv) If $f(x,d) \in \mathrm{Dom}(\mathcal{A})$ (i.e., f is of the form (2.8) with $u, v \in \mathrm{Dom}(\mathcal{A}^1)$) and (X^{ϕ}, D^{ϕ}) starts with $X_0^{\phi} = x > 0$ and $D_0^{\phi} = d \in \{0,1\}$, then the process

$$M_t^f := f(X_t^{\phi}, D_t^{\phi}) - f(x,d) - \int_0^t \mathcal{A}^{\phi} f(X_s^{\phi}, D_s^{\phi}) ds$$

is an $\mathcal{F}$-martingale.

Now we turn our attention to the semimartingale characterization of the process (X^{ϕ}, D^{ϕ}) (see Jacod and Shiryaev (2002), p.76, for the definition of predictable characteristics of a semimartingale).

Theorem 3.3. (Semimartingale characterization of (X^{ϕ}, D^{ϕ})) *(i) The bi-variate $\mathcal{F}$-semimartingale (X^{ϕ}, D^{ϕ}) has the following predictable characteristics. The predictable quadratic variation of the continuous local martingale component $X_t^{\phi,c}$ is:*

$$C_t^{X^{\phi}X^{\phi}} = \gamma \int_0^t a^2(X_s^{\phi}) ds$$

($C_t^{D^{\phi}D^{\phi}} = 0$ and $C_t^{X^{\phi}D^{\phi}} = 0$ since D^{ϕ} is purely discontinuous). The predictable process of finite variation associated with the truncation function ($h^{X^{\phi}}(x) = x\mathbb{I}_{\{|x|\leq 1\}}, h^{D^{\phi}}(x,d) = d$) is:

$$B_t^{X^{\phi}} = \int_0^t b^{\phi,0}(X_s^{\phi}) ds, \qquad B_t^{D^{\phi}} = \int_0^t (1 - D_s^{\phi}) k^{\phi}(X_s^{\phi}) ds, \tag{3.6}$$

where the function $b^{\phi,0}(x)$ is defined in Eq. (3.4) and $k^{\phi}(X_s^{\phi})$ is defined in Eq. (3.5). The compensator of the random measure $\mu^{X^{\phi},D^{\phi}}(\omega; dt, dy\, dz) = \sum_u \mathbb{I}_{\{\Delta(X_u^{\phi}, D_u^{\phi})(\omega) \neq 0\}} \delta_{(u, \Delta X_u^{\phi}(\omega), 1)}(ds, dy\, dz)$ associated to the jumps of (X^{ϕ}, D^{ϕ}) is a predictable random measure on $\mathbb{R}_+ \times (\mathbb{R}^2 \backslash \{(0,0)\})$:

$$\nu^{X^{\phi},D^{\phi}}(\omega; dt, dy\, dz) = \left[\pi^{\phi,0}(x,y) - (1-d)(\pi^{\phi,0}(x,y) - \pi^{\phi,1}(x,y)) \right] dy\, \delta_0(dz) \\ + (1-d)\gamma k(x) \delta_0(dy) \delta_1(dz) + (1-d)(\pi^{\phi,0}(x,y) - \pi^{\phi,1}(x,y)) dy\, \delta_1(dz),$$

where $\pi^{\phi,\beta}(x,y)$ *are the Lévy densities defined in Eq.* (3.3) *with* $\beta = 0,1$*, and* δ_a *is the Dirac measure charging* a*.*

(ii) The Lévy-Itô canonical representation of X^ϕ *with respect to the truncation function* $h^{X^\phi}(x) = x\mathbb{I}_{\{|x|\le 1\}}$ *is:*

$$X_t^\phi = x + B_t^{X^\phi} + X_t^{\phi,c} + \int_0^t \int_{\mathbb{R}} y\mathbb{I}_{\{|y|\le 1\}} \left(\mu^{X^\phi}(ds,dy) - \nu^{X^\phi}(ds,dy)\right) + \int_0^t \int_{\mathbb{R}} y\mathbb{I}_{\{|y|>1\}}\mu^{X^\phi}(ds,dy),$$

where the compensator of the random measure $\mu^{X^\phi}(\omega;dt,dy) = \sum_u \mathbb{I}_{\{\Delta X_u^\phi(\omega)\neq 0\}}\delta_{(u,\Delta X_u^\phi(\omega))}(ds,dy)$ *associated to the jumps of* X^ϕ *is a predictable random measure on* $\mathbb{R}_+ \times (\mathbb{R}\backslash\{0\})$*:*

$$\nu^{X^\phi}(\omega;dt,dy) = \pi^{\phi,0}(X_{t-}^\phi, y)dy\,dt. \tag{3.7}$$

(iii) The Doob-Meyer decomposition of D_t^ϕ *is:*

$$D_t^\phi = B_t^{D^\phi} + M_t^\phi \tag{3.8}$$

with the martingale $M_t^\phi = D_t^\phi - B_t^{D^\phi}$ *and the predictable compensator* $B_t^{D^\phi}$ *given in Eq.* (3.6)*, so that the* $\mathcal{F}$*-intensity is* $\lambda_t^\phi = (1-D_t^\phi)k^\phi(X_t^\phi)$*.*

Lastly, we formulate Itô formula for functions of the bi-variate process in the form convenient for our application. Since for $\rho = \phi(-\mu)$ the stock price process S is a martingale (hence, special) then it suffices to present Itô's formula for the case in which the process $f(t, X_t^\phi, D_t^\phi)$ is a special semimartingale (see. Jacod and Shiryaev (2002), Definition 4.21, p.43).

Theorem 3.4. (Itô Formula for (X^ϕ, D^ϕ)) *Suppose* (X^ϕ, D^ϕ) *starts from* $X_0^\phi = x > 0$ *and* $D_0^\phi = d \in \{0,1\}$*. For any function* $f(t,x,d) = v(t,x) + (1-d)(u(t,x)+v(t,x))$ *with* $u(t,x)$ *and* $v(t,x) \in C^{1,2}(\mathbb{R}_+ \times (0,\infty))$ *(recall that zero is an unattainable boundary for the diffusion process* X *starting at* $x > 0$*), if* $f(t, X_t^\phi, D_t^\phi)$ *is a special semimartingale, Itô formula can be written in the following form:*

$$\begin{aligned} f(t, X_t^\phi, D_t^\phi) &= f(0,x,d) + \int_0^t \left(\partial_s + \mathcal{A}^\phi\right) f(s, X_s^\phi, D_t^\phi)ds + \int_0^t \partial_x f(s, X_s^\phi, D_t^\phi)dX_s^{\phi,c} \\ &+ \int_0^t \int_{\mathbb{R}} \left(v(s, X_{s-}^\phi + y) - v(s, X_{s-}^\phi)\right)(\mu^{X^\phi}(X_{s-}^\phi; dsdy) - \nu^{X^\phi}(X_{s-}^\phi; dsdy)). \\ &+ \int_0^t \int_{\mathbb{R}} \left((u-v)(s, X_{s-}^\phi + y) - (u-v)(s, X_{s-}^\phi)\right)(1 - D_{s-}^\phi)(\hat{\mu}(X_{s-}^\phi; dsdy) - \hat{\nu}(X_{s-}^\phi; dsdy)). \\ &- \int_0^t (1 - D_{s-}^\phi)(u-v)(X_{s-}^\phi)dM_s^\phi. \end{aligned}$$

where we introduced a random measure associated to those jumps of X^ϕ *that do* not *coincide with jumps of* D^ϕ*,*

$$\hat{\mu}(\omega; ds, dy) = \sum_u \mathbb{I}_{\{\Delta X_u^\phi(\omega)\neq 0\}}\mathbb{I}_{\{\Delta D_u^\phi(\omega)=0\}}\delta_{(u,\Delta X_u^\phi(\omega))}(ds,dy), \tag{3.9}$$

and its compensator measure

$$\hat{\nu}(\omega; ds, dy) = \left[\pi^{0,\phi}(X_{s-}^\phi, y) - (1 - D_{s-}^\phi)(\pi^{0,\phi}(X_{s-}^\phi, y) - \pi^{1,\phi}(X_{s-}^\phi, y))\right] dy\,ds. \tag{3.10}$$

And where μ^{X^ϕ} *is the random measure associated to the jumps of* X^ϕ *and* ν^{X^ϕ} *is its compensator measure* (3.7)*. The generator* $\mathcal{A}^\phi$ *is given by Eq.* (3.2)*.*

Observe also that the continuous local martingale part that can be represented as $X_t^{\phi,c} = \sqrt{\gamma}\int_0^t a(X_s^\phi)dW_s$ where W is a Brownian motion, which will be convenient for our representation of the stock price process S.

4 Variance Swap Computation

Due to the semimartingale characterization of the process (X^ϕ, D^ϕ) of section 3 we are now in position to provide the characterization of the stock price process S and to explicitly compute the value of variance swap rate $K_{var} = \mathbb{E}([\log S]_t)$. First, let us recall from Eq. (2.11) that the stock price S can be seen as a function $f(t, X_t^\phi, D_t^\phi) \in C([0,\infty) \times \tilde{E})$ which is decomposed according to (2.8) with the payoff function $u(t,x) = e^{\rho t}x$ if no default occurs by time $t \geq 0$, and zero recovery $v(t,x) = 0$ if the firm defaults prior to time $t \geq 0$. The following theorem formally characterizes the stock price process.

Theorem 4.1. (Stock Price Process S) *Let the stock price process S be specified in terms of the bivariate process (X^ϕ, D^ϕ) by the prescription $S_t = e^{\rho t}(1 - D_t^\phi)X_t^\phi$, where $X_0^\phi = x > 0$ and $D_0^\phi = d \in \{0,1\}$. Moreover, assume that the scaling factor ρ satisfies $\rho = \phi(-\mu)$ where $\phi(u)$ is the Laplace exponent* (2.10) *of the subordinator T, and where $\mu \in \mathfrak{I}$ is the constant drift of the background process X* (2.2)–(2.3) *(the set $\mathfrak{I}$ is defined in Eq.* (2.9)*). Then, the stock price process S is a martingale with canonical representation*

$$S_t = S_0 + \sqrt{\gamma}\int_0^t \sigma(S_u)S_u dW_u + \int_0^t y(1 - D_{s-}^\phi)\big(\hat{\mu}(ds\,dy) - \pi^{\phi,1}(S_{s-}, y)dyds\big) - \int_0^t S_{u-}dM_u^\phi,$$

where $\gamma \geq 0$ is the drift of the Lévy subordinator T. The random measure $\hat{\mu}$ corresponds to those jumps of X^ϕ that do not *coincide with jumps of the default indicator D^ϕ (see Eq.* (3.9)*). The Lévy density $\pi^{\phi,1}(x,y)$ is defined in Eq.* (3.3)*. W is a Brownian motion and M^ϕ is the martingale* (3.8) *associated to D^ϕ.*

Proof. From the restrictions on μ and ρ, the stock price S_t is a discounted martingale (see, Mendoza-Arriaga et al. (2010), section 4). In the presence of a constant interest rate $r \geq 0$ and dividend yield $q \geq 0$, the stock price can be decomposed as $S_t = S_0 + A_t + \int_0^t S_{u-}dM_u$, where M_t is a martingale and where $A_t = \int_0^t (r-q)S_{u-}du$ is predictable, and hence, it is a special semimartingale. Then, the canonical representation of S follows from Itô's formula of Theorem 3.4 applied to the function $f(t,x,d)$ defined by $u(t,x) = e^{\rho t}x$ and $v(t,x) = 0$. We further observe that the drift vanishes since

$$\begin{aligned}
\big(\partial_s + \mathcal{A}^{\phi,1}\big)u(s, X_s^\phi, D_s^\phi) &= e^{\rho s}X_s^\phi(1 - D_s^\phi) \\
&\times \Big[\rho + \gamma[\mu + k(X_s^\phi)] + (1/X_s^\phi)\int_{(0,\infty)}\Big(\int_E (z - X_s^\phi)p^1(w, X_s^\phi, z)dz\Big)\nu(dw) \\
&- \gamma k(X_s^\phi) - \int_{(0,\infty)}\Big(1 - \int_E p^1(w, X_s^\phi, z)dz\Big)\nu(dw)\Big] \\
&= e^{\rho s}X_s^\phi(1 - D_s^\phi)\Big[\rho + \gamma\mu + \int_{(0,\infty)}\Big((1/X_s^\phi)\int_E zp^1(w, X_s^\phi, z)dz - 1\Big)\nu(dw)\Big] \\
&= e^{\rho s}X_s^\phi(1 - D_s^\phi)\Big[\rho + \gamma\mu + \int_{(0,\infty)}(e^{\mu w} - 1)\nu(dw)\Big] \\
&= e^{\rho s}X_s^\phi(1 - D_s^\phi)\big(\rho - \phi(-\mu)\big) = 0.
\end{aligned}$$

In the third equality we used the fact that $\mathbb{E}[e^{-\int_0^t k(X_u)du}X_t] = xe^{\mu t}$ (cf. Linetsky (2006), Proposition 2.1). The fourth equality follows from the definition (2.10) of the Laplace exponent, which cancels from the condition $\rho = \phi(-\mu)$. □

This canonical representation decomposes the stock price process S into a purely discontinuous martingale of jumps prior to default with the compensator measure $(1 - D_{u-}^\phi)\pi^{1,\phi}(X_{u-}^\phi, y)dydu$ (observe from Eq. (3.10) that $(1 - D_{u-}^\phi)\hat{\nu}(du, dy) = (1 - D_{u-}^\phi)\pi^{1,\phi}(X_{u-}^\phi, y)dydu$), a continuous martingale component represented in terms of a Brownian motion, and a final jump to zero (the default term $-\int_0^t S_{u-}dM_u^\phi$). Clearly, the process S is a jump-diffusion process whenever $\gamma > 0$, and a purely discontinuous process for $\gamma = 0$.

Next, we note that if the firm underlying S were to default at some time ζ^ϕ in the interval $[0,t]$, then the payoff of a traditional variance swap contract (1.1) would be infinite. To account for this possibility, we

modify the floating leg of the VS so that it only accumulates quadratic variation *prior* to the default time ζ^ϕ. That is, the long side of a VS, under our modified definition, has a payoff of

$$[\log S]_{t\wedge\zeta^\phi-} - K_{var} = \int_0^t (1-D_u^\phi) d[\log S]_u - K_{var}. \tag{4.1}$$

Notice that, for an asset that cannot default, our modified definition of a VS coincides with the traditional definition of a VS. Meanwhile, for an asset that *can* default, our modified definition of a VS is guaranteed to have a finite payoff, since the floating leg of the modified VS only accumulates quadratic variation up the time just prior to default.

Using definition (4.1), the fair value of K_{var} is the risk-neutral expectation of the floating leg

$$K_{var} = \mathbb{E}_x\left[\int_0^t (1-D_u^\phi) d[\log S]_u\right]. \tag{4.2}$$

An explicit expression for the right-hand side of (4.2) is given in the following theorem.

Theorem 4.2. *Let S be given by $S_t = e^{\rho t} X_t^\phi (1-D_t^\phi)$. Then the right-hand side of* (4.2) *is given by*

$$K_{var} = \int_0^t \mathbb{E}_x\left[\mathbb{I}_{\{\zeta^\phi>u\}}\gamma\sigma^2(X_u^\phi)\right] du + \int_0^t \mathbb{E}_x\left[\mathbb{I}_{\{\zeta^\phi>u\}}\int_{\mathbb{R}} \log^2\left(1+\frac{y}{X_{u-}^\phi}\right)\pi^{\phi,1}(X_{u-}^\phi, y)dy\right] du. \tag{4.3}$$

Proof. In view of Eq. (4.2), it suffices to first calculate the Lévy-Itô canonical representation of the function $f(t,x,d) = (1-d)\log(e^{\rho t}x)$, which corresponds to a zero–recovery log–contract on the stock price. That is, a contract that pays $u(t, X_t^\phi) = \log(e^{\rho t} X_t^\phi) = \log S_t$ if no default occurs by time $t \geq 0$, and zero otherwise (i.e., we set $v(t, X_t^\phi) = 0$). Hence, the canonical representation of the pre-default log–contract of S can be obtained by means of an application of Itô formula of Theorem 3.4 to the function $f(t,x,d) = (1-d)\log(e^{\rho t}x)$,

$$\begin{aligned} d(\log S_t) =& \big[(1-D_{t-}^\phi)\big(\partial_t + \mathcal{A}^{\phi,1}\big)\log(e^{\rho t}X_{t-}^\phi)\big]\, dt + (1-D_{t-}^\phi)\sqrt{\gamma}\,\sigma(X_{t-}^\phi)dW_t \\ &+ (1-D_{t-}^\phi)\int_{\mathbb{R}} \log\left(1+\frac{y}{X_{t-}^\phi}\right)\left(\hat{\mu}(dt\,dy) - \pi^{\phi,1}(X_{t-}^\phi, y)dydt\right) \\ &- (1-D_{t-}^\phi)\log(e^{\rho t}X_{t-}^\phi)dM_t^\phi. \end{aligned} \tag{4.4}$$

Observe that due to the default term $(1-D_{t-}^\phi)\log(e^{\rho t}X_{t-}^\phi)dM_t^\phi = (1-D_{t-}^\phi)\log(S_{t-})dM_t^\phi$ the process jumps to zero at default, which is consistent with our selection of the function $f(t,x,d) = (1-d)\log(e^{\rho t}x)$ that has zero-recovery in case of default. Consequently, it describes the pre-default dynamics of $\log S_t$ and prevents it from exploding at default time. From (4.4) it is straightforward to compute the differential $d[\log S]_t$ of the pre-default dynamics of $\log S_t$,

$$\begin{aligned} d[\log S]_t =& (1-D_{t-}^\phi)\left(\gamma\sigma^2(X_t^\phi) + \int_{\mathbb{R}} \log^2\left(1+\frac{y}{X_{t-}^\phi}\right)\pi^{\phi,1}(X_{t-}^\phi, y)dy\right) dt \\ &+ (1-D_{t-}^\phi)\int_{\mathbb{R}} \log^2\left(1+\frac{y}{X_{t-}^\phi}\right)\left(\hat{\mu}(dt\,dy) - \pi^{\phi,1}(X_{t-}^\phi, y)dydt\right) \\ &+ (1-D_{t-}^\phi)\log^2(e^{\rho t}X_{t-}^\phi)dD_t^\phi. \end{aligned} \tag{4.5}$$

Finally, multiplying (4.5) by $1-D_t^\phi = \mathbb{I}_{\{t<\zeta^\phi\}}$, observing that $(1-D_{t-}^\phi)(1-D_t^\phi) = (1-D_t^\phi)$ and $(1-D_t^\phi)dD_t^\phi = 0$ a.s., integrating over the interval $[0,t]$, taking an expectation, and using the fact that the random measure $(1-D_{t-}^\phi)\big(\hat{\mu}(dt\,dy) - \pi^{\phi,1}(X_{t-}^\phi, y)dydt\big)$ is a martingale measure, one arrives at (4.3). ☐

Next, we give an alternative formulation of the value of K_{var} in terms of Feynman-Kac semigroups.

Proposition 4.3. *Let S be given by $S_t = e^{\rho t} X_t^{\phi}(1 - D_t^{\phi})$ with $X_0 = x > 0$ and $D_0^{\phi} = d \in \{0, 1\}$. Also, let $\mathcal{P}_t^1$ (resp., $\mathcal{P}_t^{\phi,1}$) be the (resp., subordinate) Feynman-Kac semigroup defined in Eq.* (2.5) *(resp., Theorem 3.2). Then, K_{var} can be represented as follows,*

$$K_{var} = \mathbb{I}_{\{\zeta^{\phi}>0\}}\, \gamma \int_0^t (\mathcal{P}_u^{\phi,1} \sigma^2(x)) du + \mathbb{I}_{\{\zeta^{\phi}>0\}} \int_0^t \Big(\int_{(0,\infty)} \mathcal{P}_u^{\phi,1} f(s, \cdot)(x) \nu(ds) \Big) du, \tag{4.6}$$

with

$$f(s, y) = \mathcal{P}_s^1 \log^2(y) - 2\log(y) \mathcal{P}_s^1 \log(y) + \log^2(y) \mathcal{P}_s^1 1, \quad y = X_u^{\phi}.$$

Proof. From Theorem 3.2 we observe that $\mathbb{E}[(1 - D_t^{\phi}) f(X_t^{\phi})] = (1 - d)\mathcal{P}_t^{\phi,1} f(x)$. Therefore, the first term of Eq. (4.6) follows immediately. From Proposition 32.5(*iii*) in Sato (1999), p.215, we know that if $\|\mathcal{P}_t f(x) - f(x)\| = O(t)$ as $t \downarrow 0$, then $\int_{\mathbb{R}} f(y) \pi^{\phi}(x, y) dy = \int_{(0,\infty)} (\mathcal{P}_s f(x) - f(x)) \nu(ds)$. Since $f(y) = \log^2(y/x) = O(|x - y|^2)$ as $y \to x$, then to prove $\|\mathcal{P}_t^1 f(x) - f(x)\| = O(t)$ suffices to show that $\int_E (y - x)^2 p^1(t, x, y) dy = O(t)$ as $t \downarrow 0$. Indeed, the later holds true since for an arbitrary $\epsilon > 0$, we have $\int_E \mathbb{I}_{\{|x-y|<\epsilon\}} (y - x)^2 p^1(t, x, y) dy \le Ct$ as $t \downarrow 0$ (cf., McKean (1956), Theorem 4.5). Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}} \log^2 \Big(1 + \frac{y}{X_{u-}^{\phi}}\Big) \pi^{\phi,1}(X_{u-}^{\phi}, y) dy \\
&= \int_{(0,\infty)} \Big(\int_{E \setminus \{x\}} \log^2 \Big(\frac{y}{X_{u-}^{\phi}}\Big) p^1(s, X_{u-}^{\phi}, y) dy \Big) \nu(ds) \\
&= \int_{(0,\infty)} \big(\mathcal{P}_s^1 \log^2(X_{u-}) - 2\log(X_{u-}) \mathcal{P}_s^1 \log(X_{u-}) + \log^2(X_{u-}) \mathcal{P}_s^1 1 \big) \nu(ds).
\end{aligned}$$

The rest follows from observing that $\mathbb{E}[(1 - D_u^{\phi}) f(s, X_u^{\phi})] = (1 - d)\mathcal{P}_u^{\phi,1} f(s, x)$. □

5 Spectral Expansions

In order for the results of sections 3 and 4 to be useful, we need a practical way to construct the FK semigroups $\{\mathcal{P}_t^1, t \ge 0\}$ and $\{\mathcal{P}_t^{\phi,1}, t \ge 0\}$ as well as the associated transition densities $p^1(t, x, y)$ and $p^{\phi,1}(t, x, y)$ (recall that we had dropped the super-index α, since we only need the case of $\alpha = 1$). Spectral theory, or more specifically, the theory of eigenfunction expansions, provides a straightforward method of constructing these operators and functions. Below, we review some useful results relating to eigenfunction expansions. A detailed description of the spectral Theorem for self-adjoint operators in a Hilbert space is given in appendix B.

Recall that the FK semigroup $\{\mathcal{P}_t^1, t \ge 0\}$ has infinitesimal generator $\mathcal{A}^1$ (2.6). With $\mathcal{A}^1$ we associate a *scale density* $\mathfrak{s}$ and *speed density* $\mathfrak{m}$

$$\mathfrak{s}(x) := \exp\left(-\int_{x_0}^{x} \frac{2\, b(y)}{a^2(y)} dy \right), \qquad \mathfrak{m}(x) := \frac{2}{a^2(x)} \exp\left(\int_{x_0}^{x} \frac{2\, b(y)}{a^2(y)} dy \right),$$

where the point x_0 is arbitrarily chosen in $E = (0, \infty)$. The generator $\mathcal{A}^1$ (2.6) with domain

$$\mathrm{Dom}(\mathcal{A}^1) = \{ f \in L^2(E, \mathfrak{m}) : \mathcal{A}^1 f \in L^2(E, \mathfrak{m}) \},$$

is a self-adjoint operator in the Hilbert space $\mathcal{H} = L^2(E, \mathfrak{m})$. [3] Therefore, we have spectral representations for the operators $\mathcal{A}^1$ and $g(\mathcal{A}^1)$, where g is any Borel-measurable function.

[3] $\mathcal{A}^1$ is dense in $\mathcal{H}$ implies that $\mathcal{A}^1$ has a unique self-adjoint extension $\overline{\mathcal{A}^1}$ with $\mathrm{Dom}(\overline{\mathcal{A}^1}) = \mathcal{H}$. We will not distinguish between $\mathcal{A}^1$ and its extension $\overline{\mathcal{A}^1}$.

Let ψ_λ and λ be the generalized eigenfunctions/values of $-\mathcal{A}^1$. Note that, since $\mathcal{A}^1$ is the generator of a contraction semigroup $\{\mathcal{P}^1_t, t \geq 0\}$, the eigenvalues of $-\mathcal{A}^1$ are non-negative. The operator $\mathcal{P}^1_t$ can be written as $\mathcal{P}^1_t = e^{t\mathcal{A}^1}$ (for a general Banach space, when the generator $\mathcal{A}^1$ is unbounded the latter is understood as a strong limit via the Yosida approximation (see Pazy (1983), Corollary 3.5)). Thus, using (B.2), for any $f \in \mathcal{H}$ we have

$$\mathcal{P}^1_t f(x) = \sum_\lambda e^{-\lambda t} c_\lambda \psi_\lambda(x), \qquad c_\lambda = (\psi_\lambda, f), \qquad (\psi_\lambda, f) = \int_E \overline{\psi_\lambda}(x) f(x) \mathfrak{m}(x) dx, \tag{5.1}$$

where $\overline{\psi_\lambda}$ indicates the complex conjugate of ψ_λ. The notation $\sum_\lambda(\cdots)$ is shorthand for

$$\sum_\lambda (\cdots) = \sum_{\lambda_n \in \sigma_d(-\mathcal{A}^1)} (\cdots) + \int_{\lambda_\omega \in \sigma_c(-\mathcal{A}^1)} (\cdots)\, d\omega,$$

where $\sigma_d(-\mathcal{A}^1)$ and $\sigma_c(-\mathcal{A}^1)$ are the discrete and continuous portions of the spectrum of $-\mathcal{A}^1$ respectively.

Similarly, using the functional calculus of Theorem B.1, the subordinated semigroup $\mathcal{P}^{\phi,1}_t$ defined in Eq. (3.1) can be obtained as,

$$\mathcal{P}^{\phi,1}_t f(x) = \sum_\lambda e^{-\phi(\lambda) t} c_\lambda \psi_\lambda(x), \qquad c_\lambda = (\psi_\lambda, f), \qquad (\psi_\lambda, f) = \int_E \overline{\psi_\lambda}(x) f(x) \mathfrak{m}(x) dx,$$

where $\phi(\lambda)$ is the Laplace exponent of T, defined in Eq. (2.10). One should mention that the recent book of Schilling et al. (2010) is an excellent reference for Bochner subordination of semigroups (for example, the last result above is obtained from their Remark 12.4, p.133).

5.1 Uniformly convergence of the discrete spectrum

In general, when the spectrum is discrete, the spectral expansion (5.1) of the semigroup $\mathcal{P}^1$ (and hence, $\mathcal{P}^{\phi,1}$) leads to an infinite series. When the semigroup $\mathcal{P}^1$ is of trace class, then it is possible to establish uniform convergence for the expansions as follows. Assume that the eigenvalues of $-\mathcal{A}^1$ satisfy the condition

$$\sum_\lambda e^{-\lambda t} < \infty, \qquad \forall\, t > 0, \tag{5.2}$$

so that the FK semigroup $\{\mathcal{P}^1_t, t \geq 0\}$ is *trace class* (see Section 7.2 of Davies (2007)). According to Theorem 7.2.5 of Davies (2007), if $\{\mathcal{P}^1_t, t \geq 0\}$ is trace class, then the eigenfunctions $\psi_\lambda(x)$ are continuous functions with the global estimate $|\psi_\lambda(x)| \leq e^{\lambda t/2}\sqrt{p^1(t,x,x)/\mathfrak{m}(x)}$ for all $t \geq 0$. Setting $f(x) = \delta_y(x)$ in (5.1) yields the transition density $p^1(t,x,y)$ of the FK semigroup.

$$p^1(t,x,y) = \mathfrak{m}(y) \sum_\lambda e^{-\lambda t} \psi_\lambda(x) \overline{\psi}_\lambda(y). \tag{5.3}$$

The sum on the right-hand side of (5.3) converges uniformly in x and y on compacts. This ensures that, in addition to the L^2 convergence, the eigenfunction expansion (5.1) converges uniformly in x on compacts for all $f \in L^2(E, \mathfrak{m})$ and $t > 0$. This follows from the Cauchy-Schwarz bound for the expansion coefficients $|c_\lambda| \leq \sqrt{(f,f)}$, the eigenfunction estimate, and the trace class condition (5.2).

In this case, the spectral expansion for the subordinated FK semigroup $\{\mathcal{P}^{\phi,1}_t, t \geq 0\}$ can be obtained by conditioning on the subordinator T_t. For any $f \in \mathcal{H}$ we have

$$\mathcal{P}^{\phi,1}_t f(x) = \mathbb{E}_x\left[\mathcal{P}^1_{T_t} f(x)\right] = \mathbb{E}\left[\mathbb{E}_x\left[\mathcal{P}^1_{T_t} f(x) | T_t\right]\right] = \sum_\lambda \mathbb{E}\left[e^{-\lambda T_t}\right] c_\lambda \psi_\lambda(x) = \sum_\lambda e^{-\phi(\lambda) t} c_\lambda \psi_\lambda(x), \tag{5.4}$$

where $c_\lambda = (\psi_\lambda, f)$ and $\phi(\lambda)$ is the Lévy exponent of the subordinator T. If we assume that the Laplace exponent ϕ is such that

$$\sum_\lambda e^{-\phi(\lambda)t} < \infty, \qquad \forall\, t > 0,$$

then the subordinated FK semigroup $\{\mathcal{P}_t^{\phi,1}, t \geq 0\}$ is trace class. If we further assume that the eigenfunctions ψ_λ of the FK semigroup $\{\mathcal{P}_t^1, t \geq 0\}$ have a bound independent of λ on each compact interval $K = [a, b] \subset E$ (that is, if there exist constants C_K, which depend on the compact interval K but are *independent* of λ, such that $|\psi_\lambda(x)| \leq C_K$ for all $\lambda \in \sigma_d(-\mathcal{A}^1)$) then, in addition to the L^2 convergence, the eigenfunction expansion of the subordinated FK semigroup (5.4) converges uniformly in x on compacts for all $f \in L^2(E, \mathfrak{m})$ and $t > 0$. As above, setting $f(x) = \delta_y(x)$ in (5.4) yields the transition density $p^{\phi,1}(t, x, y)$ of the subordinated FK semigroup

$$p^{\phi,1}(t, x, y) = \mathfrak{m}(y) \sum_\lambda e^{-\phi(\lambda)t} \psi_\lambda(x) \overline{\psi}_\lambda(y). \tag{5.5}$$

The sum in (5.5) is uniformly convergent on compacts in x and y. Note that the semigroup corresponding to the JDCEV process of section 6.2 is of trace class.

6 Examples

In this section, we compute K_{var} (4.3) explicitly (up to an integral with respect to the Lévy measure ν of the subordinator T), when the background Feller diffusion X is modeled as (i) a geometric Brownian motion with constant killing rate and (ii) a Jump-to-default Constant Elasticity of Variance process.

6.1 Example: geometric Brownian motion with default

Perhaps the most widely recognized non-negative diffusion in finance is the geometric Brownian motion process (GBM). Here, we consider GBM with a constant killing rate, which is a diffusion of the form (2.2)-(2.3) with constant parameters $k(x) = k \geq 0$ and $\sigma(x) = \sigma > 0$ (excuse the abuse of notation). The generator $\mathcal{A}^1$ of the FK semigroup $\{\mathcal{P}_t^1, t \geq 0\}$ and the corresponding speed density $\mathfrak{m}(x)$ are given by

$$\mathcal{A}^1 = \tfrac{1}{2}\sigma^2 x^2\, \partial_{xx}^2 + (\mu + k) x\, \partial_x - k, \tag{6.1}$$

$$\mathfrak{m}(x) = \frac{2}{x\sigma^2} \exp\left(2\,\xi \frac{1}{\sigma} \log x\right), \qquad \xi = \frac{\mu + k}{\sigma} - \frac{\sigma}{2}. \tag{6.2}$$

Most commonly, the FK transition density $p^1(t, x, y)$ of GBM with default is written as

$$p^1(t, x, y) = \frac{e^{-kt}}{x\sigma\sqrt{2\pi t}} \exp\left(\frac{-\left(\log y - \log x - (\mu + k - \sigma^2/2)t\right)^2}{2\sigma^2 t}\right). \tag{6.3}$$

Due to the fact that $\mathcal{A}^1$ (6.1) is a self-adjoint operator on the Hilbert space $\mathcal{H} = L^2(E, \mathfrak{m})$, with $\mathfrak{m}$ given by (6.2), the FK transition density $p^1(t, x, y)$ also has an (generalized) eigenfunction expansion of the form (5.3). The eigenfunctions and eigenvalues of $-\mathcal{A}^1$ are given in the following Theorem.

Theorem 6.1 (GBM Eigenvalues and Eigenfunctions)**.** *Let the operator* $\mathcal{A}^1$ *be given by* (6.1). *The spectrum of* $\mathcal{A}^1$ *is purely continuous:* $\sigma(\mathcal{A}^1) = \sigma_c(\mathcal{A}^1)$. *The improper eigenfunctions of* $-\mathcal{A}^1$ *and the corresponding improper eigenvalues are*

$$\psi_\omega(x) = \sqrt{\frac{\sigma}{4\pi}} \exp\left((i\omega - \xi)\frac{1}{\sigma} \log x\right), \qquad \lambda_\omega = \frac{1}{2}\left(\omega^2 + \xi^2\right) + k, \qquad \omega \in (-\infty, \infty), \tag{6.4}$$

where ξ *is given in* (6.2).

Proof. One can check directly that the eigenfunctions and eigenvalues of (6.4) satisfy the improper eigenvalue equation $-\mathcal{A}^1\psi_\omega = \lambda_\omega\psi_\omega$ and the boundedness condition (B.3). The orthogonality relation $(\psi_\omega, \psi_\nu) = \delta(\omega-\nu)$ follows by noting that $\frac{1}{2\pi}\int e^{-i(\omega-\nu)x}dx = \delta(\omega-\nu)$. □

Remark 6.2. Transition density (6.3) can be obtained by writing eigenfunction expansion (5.3) with the eigenfunctions and eigenvalues of (6.4), making a change of variables $z = \frac{1}{\sigma}\log y$, and using the Fourier transform of a Gaussian density

$$\frac{1}{\sqrt{2\pi a^2}}\exp\left(\frac{-(z-b)^2}{2a^2}\right) = \int_{(-\infty,\infty)} \frac{1}{2\pi}\exp\left(i\,(b-z)\,\omega - \frac{a^2\omega^2}{2}\right)d\omega.$$

Remark 6.3. When the underlying S is given by (2.1) and the background diffusion X is modeled as a GBM with default, the at-the-money skew of the model-induced implied volatility surface is controlled by μ. For $\mu < 0$ jumps in S will be preferentially downward, causing a negative at-the-money skew. For $\mu > 0$, jumps in S will be preferentially upward, causing a positive at-the-money skew. As the skew for equity options is typically negative, it makes sense to choose $\mu < 0$.

We are now in position to compute (4.3) when X is modeled as a GBM with default.

Proposition 6.4. *Let X be a GBM process with default as described above. Then we have (i)*

$$\int_0^t \mathbb{E}_x\left[\mathbb{I}_{\{\zeta^\phi > u\}}\gamma\sigma^2(X_u^\phi)\right]du = \gamma\,\sigma^2\left(\frac{1-e^{-kt}}{k}\right),$$

and (ii)

$$\int_0^t \mathbb{E}_x\left[\mathbb{I}_{\{\zeta^\phi > u\}}\int_{\mathbb{R}}\log^2\left(1+\frac{y}{X_{u-}^\phi}\right)\pi^{\phi,1}(X_{u-}^\phi, y)dy\right]du = \sigma^2\left(\frac{1-e^{-\phi(k)t}}{\phi(k)}\right)\int_{(0,\infty)} e^{-ks}\left(s+s^2\xi^2\right)\nu(ds).$$

Proof. See Appendix A.1. □

6.2 Example: Jump-to-Default constant elasticity of variance

The Constant Elasticity of Variance (CEV) model of Cox (1975) is a non-negative diffusion of the form (2.2)-(2.3), where $k(x) = 0$ and

$$\sigma(x) = ax^\beta.$$

Here, $\beta < 0$ is the volatility elasticity parameter and $a > 0$ is the volatility scale parameter. The specification $\beta < 0$ is consistent with the leverage effect (volatility increases when the stock price falls). For $\beta < 0$ the CEV process hits zero with positive probability. In particular, for $\beta \in [-1/2, 0)$, the origin is an exit boundary. For $\beta < -1/2$ the origin is a regular boundary specified as a killing boundary.

Carr and Linetsky (2006) extend the CEV model to include a possible jump-to-default. Their model is refereed to as jump-to-default CEV or, more succinctly, JDCEV. In the JDCEV framework, the jump to default has a killing rate which is an affine functions of the local variance

$$k(x) = b + c\sigma^2(x) = b + ca^2x^{2\beta},$$

where $b \geq 0$ and $c \geq 0$. Although for all $c > 0$ default may only occur through a jump from a positive value. When $c \geq 1/2$ the zero boundary is *entrance* for the JDCEV diffusion, and thus, the diffusion cannot reach zero from the interior of E. The majority of the expressions developed in this section hold for all $c > 0$. However, *the credit-equity modeling framework developed in Sections 2–4 works exclusively for the case in which* $c \geq 1/2$ (i.e., the case in which zero is an entrance boundary). Therefore, one should keep in mind this restriction when applying the following more general results.

For a JDCEV diffusion, the generator $\mathcal{A}^1$ of the FK semigroup $\mathcal{P}^1$ and the corresponding speed density are given by

$$\mathcal{A}^1 = \tfrac{1}{2}a^2x^{2\beta+2}\,\partial^2_{xx} + \left(\mu + b + ca^2x^{2\beta}\right)x\,\partial_x - \left(b + ca^2x^{2\beta}\right). \tag{6.5}$$

$$\mathfrak{m}(x) = \frac{2}{a^2}x^{2c-2-2\beta}\exp\left(\varepsilon\,A\,x^{-2\beta}\right), \qquad A = \frac{|\mu+b|}{a^2|\beta|}, \qquad \varepsilon = \mathrm{sign}(\mu+b). \tag{6.6}$$

The FK transition density $p^1(t,x,y)$ for the JDCEV diffusion was obtained by Carr and Linetsky (2006)

$$p^1(t,x,y) = \frac{\mathfrak{m}(x)|\mu+b|(xy)^{\frac{1}{2}-c}e^{\omega\nu t/2}}{1-e^{-\omega t}}\exp\left(-\varepsilon A\frac{x^{-2\beta}+y^{-2\beta}}{1-e^{-\varepsilon\omega t}} - \lambda_1 t\right)I_\nu\left(\frac{A(xy)^{-\beta}}{\sinh(\omega t/2)}\right), \tag{6.7}$$

where I_ν is the modified Bessel function of order ν, the constants A and ε are given in (6.6) and

$$\nu = \frac{1+2c}{2|\beta|}, \qquad \omega = 2|\beta(\mu+b)|, \qquad \lambda_1 = \begin{cases} 2(\mu+b)(|\beta|+c)+b, & \mu+b>0 \\ |\mu|, & \mu+b<0 \end{cases}. \tag{6.8}$$

Due to the fact that the unique extension of $\mathcal{A}^1$ is a self-adjoint operator in the Hilbert space $\mathcal{H} = L^2(E,\mathfrak{m})$ with $\mathfrak{m}(x)$ given by (6.6), the FK transition density (6.7) has an eigenfunction expansion of the form (5.3). The eigenfunctions and eigenvalues of $-\mathcal{A}^1$ are given in the following theorem, which is due to Mendoza-Arriaga and Linetsky (2010).

Theorem 6.5 (JDCEV Eigenvalues and Eigenfunctions)**.** *Let $\mathcal{A}^1$ be given by* (6.5)*. When $|\mu+b| \neq 0$, the spectrum of $\mathcal{A}^1$ is purely discrete: $\sigma(\mathcal{A}^1) = \sigma_d(\mathcal{A}^1)$. The eigenfunctions of $-\mathcal{A}^1$ and the corresponding eigenvalues are:*

$$\psi_n(x) = A^{\nu/2}\sqrt{\frac{(n+1)!|\mu+b|}{\Gamma(\nu+n)}}\,x\exp\left(-\tfrac{1}{2}(1+\varepsilon)Ax^{-2\beta}\right)L^\nu_{n-1}(Ax^{-2\beta}), \tag{6.9}$$

$$\lambda_n = \omega(n-1)+\lambda_1, \qquad n = 1,2,3,\cdots, \tag{6.10}$$

where L^ν_n are the generalized Laguerre polynomials, A and ε are given in (6.6) *and ν, ω and λ_1 are given in* (6.8).

Proof. One can verify directly that the eigenfunctions and eigenvalues (6.9)-(6.10) satisfy the (proper) eigenvalues equation $-\mathcal{A}^1\psi_n = \lambda_n\psi_n$. Orthogonality of the eigenfunctions $(\psi_n,\psi_m) = \delta_{n,m}$ follows from the orthogonality relations of the generalized Laguerre polynomials (see Abramowitz and Stegun (1972), pp. 775)

$$\int_0^\infty x^\alpha e^{-x}L^\alpha_n(x)L^\alpha_m(x)dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m}.$$

$\square$

Remark 6.6. The transition density (6.7) can be recovered from the eigenfunction expansion (5.3) with the eigenfunctions and eigenvalues defined in (6.9)-(6.10) by means of the Hille-Hardy formula (see Erdelyi (1953), p.189):

$$\sum_{n=0}^\infty \frac{t^n n!}{\Gamma(n+\nu+1)}L^\nu_n(a)L^\nu_n(b) = \frac{(abt)^{-\nu/2}}{1-t}\exp\left(-\frac{(a+b)t}{1-t}\right)I_\nu\left(\frac{2\sqrt{abt}}{1-t}\right),$$

which is valid for all $|t|<1$, $\nu>-1$ and $a,b>0$.

We are now equipped to compute K_{var} (4.3) when the background process X is a JDCEV diffusion. We shall focus specifically on the case $\mu + b < 0$, since in this case, all of the relevant functions f are in $L^2(E, \mathfrak{m})$ with $\mathfrak{m}(x)$ given by (6.6). Thus, we can compute all the necessary expectations explicitly using the eigenfunction expansion techniques of section 5.

Before computing the expectations in (4.3) it will be useful to give an analytical solution for the p-th moment of the stock price

$$\mathbb{E}_x\Big[(S_t)^p\Big] = e^{p\rho t}\,\mathbb{E}_x\Big[(X_t^\phi)^p \mathbb{I}_{\{\zeta^\phi > t\}}\Big] = e^{p\rho t}\,\mathcal{P}_t^{\phi,1} x^p.$$

Proposition 6.7 (p-th Moment)**.** *Let the diffusion $\{X_t, t \geq 0\}$ be a JDCEV process with parameters $\beta < 0$, $a > 0$, $b \geq 0$, and $c > 0$. Assume $\mu + b < 0$. Then, (i) for $p > 2(\beta - c)$, the expected value of the function $f(x) = x^p$ is given by the eigenfunction expansion:*

$$\mathbb{E}_x\Big[(S_t)^p\Big] = e^{p\rho t} \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t} \tilde{c}_n \psi_n(x),$$

where $\phi(\lambda)$ is the Laplace exponent of the subordinator T. The JDCEV eigenvalues λ_n and eigenfunctions ψ_n, are given in theorem 6.5, and the expansion coefficients are given by:

$$\tilde{c}_n = (x^p, \psi_n) = \frac{A^{\frac{\nu}{2} - \frac{p+2c}{2|\beta|}} \left(\frac{1-p}{2|\beta|}\right)_{n-1}}{\sqrt{(n-1)!|\mu + b|\Gamma(\nu + n)}} \Gamma\left(\frac{p+2c}{2|\beta|} + 1\right), \quad n = 1, 2, \cdots, \tag{6.11}$$

where $(z)_n = z(z-1)\cdots(z-n-1)$ is the Pochhammer symbol. Also, (ii) the spectral expansion is uniformly convergent for all $t > 0$, absolutely convergent at $t = 0$, and uniformly convergent at $t = 0$ if $p > (\beta+1)/2 - c$.

Proof. The proof of part (i) is obtained from Lemma 3.1 and Proposition 3.1 in Mendoza-Arriaga and Linetsky (2010). For part (ii) we note that the semigroup is of trace class. In addition, we note that for $x \in [a, b] \subset E$, the eigenfunctions $\psi_n(x)$ satisfy the bound $|\psi_n(x)| < C/n^{1/4} < C$ for some $C < \infty$ independent of n although it may depend on the range $[a, b]$ (see inequality (27a) on p.54 of Nikiforov and Uvarov (1988)). Moreover, since the expansion coefficients satisfy the Cauchy-Schwartz bound, $|c_n| \leq \sqrt{(f, f)}$, then for any $f \in L^2((0, \infty), \mathfrak{m})$ the spectral expansion of $\mathcal{P}_t^{\phi,1} f$ converges uniformly for all $t > 0$. That is, $\sum_{n=1}^{\infty} e^{-\lambda_n t}|c_n \psi_n(x)| \leq C\sqrt{(f, f)} \sum_{n=1}^{\infty} e^{-\lambda_n t}$ converges uniformly for all $t > 0$. In Appendix A.2 we show that the spectral expansion of $\mathcal{P}_t^{\phi,1} x^p$ also converges absolutely at $t = 0$. In addition, if $p > (\beta + 1)/2 - c$, then the spectral expansion converges uniformly at $t = 0$. □

Proposition 6.8. *Let X be a JDCEV process with parameters $\beta < 0$, $a > 0$, $b \geq 0$, and $c > 0$. Assume $\mu + b < 0$. Then we have*

$$\begin{aligned}\gamma &\int_0^t \mathbb{E}_x\Big[\mathbb{I}_{\{\zeta^\phi > u\}} \sigma^2(X_u^\phi)\Big] du \\ &= \gamma a^2 A^{\frac{\nu}{2} - \frac{c}{|\beta|} + 1} \Gamma\left(c/|\beta|\right) \sum_{n=1}^{\infty} \frac{(1/(2|\beta|) + 1)_{n-1}}{\sqrt{(n-1)!|\mu + b|\Gamma(\nu + n)}} \frac{\left(1 - e^{-\phi(\lambda_n)t}\right)\psi_n(x)}{\phi(\lambda_n)}.\end{aligned} \tag{6.12}$$

Proof. The expectation in (6.12) can be written explicitly as

$$\begin{aligned}\gamma a^2 &\int_0^t \mathbb{E}_x\Big[\mathbb{I}_{\{\zeta^\phi > u\}} (X_u^\phi)^{2\beta}\Big] du \\ &= \gamma a^2 A^{\frac{\nu}{2} - \frac{c}{|\beta|} + 1} \Gamma\left(c/|\beta|\right) \int_0^t \Big(\sum_{n=1}^{\infty} e^{-\phi(\lambda_n)u} \frac{(1/(2|\beta|) + 1)_{n-1}}{\sqrt{(n-1)!|\mu + b|\Gamma(\nu + n)}} \psi_n(x)\Big) du,\end{aligned}$$

where the last equality is due to Proposition 6.7. One should note that if $3\beta+2c>1$ then the sum converges uniformly at $t=0$ and the integral can be done term by term. Otherwise, observe that: (a) the series inside the integral is absolutely convergent for all $t\geq 0$ due to Proposition 6.7 (and continuous for all $t\geq 0$), and (b) the Laplace exponent $\phi(\lambda)$ is increasing. Then, we can conclude that the resulting series (6.12) is also absolutely convergent, and hence the exchange of sum and integral is justified (i.e., we integrate term by term with $\int_\epsilon^t \cdot ds$ for some $\epsilon>0$ and then take the limit as $\epsilon \downarrow 0$). ☐

Proposition 6.9. *Let X be a JDCEV process with parameters $\beta<0$, $a>0$, $b\geq 0$, and $c>0$. Assume $\mu+b<0$ and $2c-2\beta>1$. Then we have*

$$
\begin{aligned}
&\int_0^t \mathbb{E}_x\left[\mathbb{I}_{\{\zeta^\phi>u\}}\int_{\mathbb{R}}\log^2\Big(1+\frac{y}{X^\phi_{u-}}\Big)\pi^{\phi,1}(X^\phi_{u-},y)dy\right]du \\
&\quad= \frac{A^{\frac{1-2c}{4|\beta|}}}{4|\beta|^2}\int_{\mathbb{R}_+\setminus\{0\}}\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{e^{-\lambda_n s}(1-e^{-\phi(\lambda_m)t})}{\phi(\lambda_m)}\psi_m(x)\times \\
&\qquad\times\Bigg\{\sqrt{\frac{(n-1)!}{|\mu+b|\Gamma(\nu+n)}}\Theta^2_{n-1}\Big(\frac{c+|\beta|}{|\beta|}\Big) \\
&\qquad\quad -2\sqrt{\frac{(m-1)!}{|\mu+b|\Gamma(\nu+m)}}\sum_{k=0}^{n-1}\frac{(1-n)_k\Theta^1_{m-1}(\nu+k+1)\Theta^1_{n-1}\Big(\frac{c+|\beta|}{|\beta|}\Big)}{\Gamma(\nu+k+1)k!} \\
&\qquad\quad +\frac{\Big(\frac{1}{2|\beta|}\Big)_{n-1}\Gamma\Big(\frac{c}{|\beta|}+1\Big)}{(n-1)!}\sqrt{\frac{(m-1)!}{|\mu+b|\Gamma(\nu+m)}}\sum_{k=0}^{n-1}\frac{(1-n)_k\Theta^2_{m-1}(\nu+k+1)}{\Gamma(\nu+k+1)k!}\Bigg\}\nu(ds),
\end{aligned}
\tag{6.13}
$$

where

$$\Theta^1_n(\alpha) := \frac{(1-\alpha+\nu)_n}{n!}\Gamma(\alpha)\Big[\psi(\alpha)-\sum_{p=1}^n\frac{1}{(p-\alpha+\nu)}\Big], \tag{6.14}$$

$$\Theta^2_n(\alpha) := \frac{(1-\alpha+\nu)_n}{n!}\Gamma(\alpha)\Big[\Big(\psi(\alpha)-\sum_{p=1}^n\frac{1}{p-\alpha+\nu}\Big)^2+\psi'(\alpha)-\sum_{p=1}^n\frac{1}{(p-\alpha+\nu)^2}\Big], \tag{6.15}$$

and $\psi(\alpha)=\Gamma'(\alpha)/\Gamma(\alpha)$ (with no subscript on ψ) is the Polygamma function.

Proof. We start by mentioning that the functions Θ^δ_n result from the integrals

$$\Theta^\delta_n(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}\log^\delta(z)L^\nu_n(z)dz, \qquad \delta\in\{1,2\}, \qquad \mathrm{Re}(\alpha)>0, \tag{6.16}$$

which are available in Prudnikov et al. (1983), Eq. 2.19.6.1 and 2.19.6.3, pp.469. The restriction $2c-2\beta>1$ is imposed such that the sums converge absolutely at $t=0$. Indeed, it is easy to show that for all $x>0$ we have $|\log(x)|\leq(1/x+x)$ and $\log^2(x)\leq(1/x+x)$. Moreover, since the expansion of $(1/x+x)$ converges absolutely at $t=0$ due to Proposition 6.7, then each of the series also converge absolutely at $t=0$. The rest of the proof consists of integrating term by term. Details are found in Appendix A.3. ☐

7 Market Implied Lévy Subordinators

Note that the value of K_{var} (4.3) depends on the drift γ and Lévy measure ν of the Lévy subordinator T. One could, of course, compute the value of K_{var} by choosing a specific drift γ and Lévy measure ν. However, this parametric approach would lead to a considerable amount of model misspecification risk, as there is no

guarantee that the chosen subordinator would induce European option prices consistent with those observed on the market. An alternative approach would be to use knowledge of (liquidly traded and efficiently priced) European call and put options to constrain one's choice of Lévy subordinator. In this section we will show that, when the background diffusion X is fixed, the drift γ and Lévy measure ν of the subordinator T can be obtained *non-parametrically* from the implied volatility smile of t-expiry European options. This approach greatly reduces model misspecification risk, as the obtained subordinator induces t-expiry option prices that are consistent with those observed on the market.

Let S be described by (2.1). We assume that background diffusion X is given, but that the drift γ and Lévy measure ν of the Lévy subordinator T are unknown. Denote by $C(t,x;K)$ the price of a European call option with time to maturity t and strike price K. Note that the price of a call with strike price K can be obtained from the price of a put with the same strike through put-call parity. We assume the existence of European call options at all strikes $K \in (0,\infty)$. While calls at all strikes $K \in (0,\infty)$ do not trade in practice, Bondarenko (2003) shows how to estimate the value of call at any strike, given the value of calls at a discrete set of strikes.

Let $p_S(t,x,y)$ be the transition density of S under the risk-neutral pricing measure

$$p_S(t,x,y)dy = \mathbb{P}_x\left[S_t \in dy\right].$$

Note that $S_t \in dy$ if and only if $(1-D_t^{\phi})e^{\rho t}X_t^{\phi} \in dy$. Thus,

$$p_S(t,x,y) = e^{-\rho t}p^{\phi,1}(t,x,y') = e^{-\rho t}\mathfrak{m}(y')\sum_{\lambda} e^{-\phi(\lambda)t}\psi_\lambda(x)\overline{\psi}_\lambda(y'), \qquad y' := y\,e^{-\rho t}. \tag{7.1}$$

As Breeden and Litzenberger (1978) show, the transition density $p_S(t,x,y)$ can be implied from a semi-infinite strip of call prices. We have

$$C(t,x;K) = \mathbb{E}_x\left[(S_t-K)^+\right] = \int_E (y-K)^+ p_S(t,x,y)dy. \tag{7.2}$$

Differentiating both sides of (7.2) twice with respect to K, and noting that $\partial^2_{KK}(y-K)^+ = \delta(y-K)$, one obtains

$$\partial^2_{KK}C(t,x;K) = \int_E \partial^2_{KK}(y-K)^+ p_S(t,x,y)dy = p_S(t,x,K). \tag{7.3}$$

Setting our the two expressions (7.1) and (7.3) for p_S equal to each other yields

$$\partial^2_{KK}C(t,x;K) = e^{-\rho t}\mathfrak{m}(K')\sum_{\lambda} e^{-\phi(\lambda)t}\psi_\lambda(x)\overline{\psi}_\lambda(K'), \qquad K' := K\,e^{-\rho t}. \tag{7.4}$$

Multiplying both sides of (7.4) by $\psi_{\lambda'}(K')$ and integrating with respect to K, we obtain

$$\begin{aligned}\int_E \partial^2_{KK}C(t,x;K)\overline{\psi}_{\lambda'}(K')dK &= e^{-\rho t}\sum_{\lambda} e^{-\phi(\lambda)t}\psi_\lambda(x)\int_E \overline{\psi}_\lambda(K')\psi_{\lambda'}(K')\mathfrak{m}(K')dK \\ &= \sum_{\lambda} e^{-\phi(\lambda)t}\psi_\lambda(x)\left(\psi_\lambda,\psi_{\lambda'}\right) = e^{-\phi(\lambda')t}\psi_{\lambda'}(x).\end{aligned} \tag{7.5}$$

Note that we have used $dK' = e^{-\rho t}dK$ and $(\psi_\lambda,\psi_{\lambda'}) = \delta_{\lambda,\lambda'}$. We solve (7.5) for $\phi(\lambda)$

$$\phi(\lambda) = \frac{-1}{t}\log\left(\frac{\int \partial^2_{KK}C(t,x;K)\psi_\lambda(K')dK}{\psi_\lambda(x)}\right). \tag{7.6}$$

If there exists a Lévy subordinator T independent of X, which is capable of generating the prices of call options on the market, then its Laplace exponent, evaluated at λ is given by (7.6). [4] Thus, we refer to T with Laplace exponent (7.6) as the *market implied Lévy subordinator*.

[4] In a working paper, Carr and Lee (2006) obtain $\mathbb{E}\left[e^{-\lambda T_t}\right]$ using similar methods. The authors do not deal with Lévy subordinators specifically.

Remark 7.1. It is worth noting that, although one can theoretically compute $\partial^2_{KK}C(t,x;K)$ with call prices available at all strikes $K \in (0,\infty)$, a more convenient expression for the integral in (7.6) can be obtained by integrating by parts twice [5]

$$\int_0^\infty \partial^2_{KK}C(t,x;K)\psi_\lambda(K')dK = \partial_K C(t,x;K)\psi_\lambda(K')\Big|_0^\infty - C(t,x;K)\partial_K\psi_\lambda(K')\Big|_0^\infty + \int_0^\infty C(t,x;K)\partial^2_{KK}\psi_\lambda(K')dK.$$

We have the following limits

$$\lim_{K\to\infty} \partial_K C(t,x;K) = 0, \qquad \lim_{K\to\infty} C(t,x;K) = 0,$$
$$\lim_{K\to 0} \partial_K C(t,x;K) = -1, \qquad \lim_{K\to 0} C(t,x;K) = x.$$

Hence, we find

$$\int_0^\infty \partial^2_{KK}C(t,x;K)\psi_n(K')dK = \psi_\lambda(0) + x\,e^{-\rho t}\partial_x\psi_\lambda(0) + \int_0^\infty C(t,x;K)\partial^2_{KK}\psi_\lambda(K')dK. \tag{7.7}$$

Note that ∂_x is a derivative with respect to the argument of ψ_λ whereas ∂^2_{KK} is a derivative with respect to K. The advantage of using expression (7.7) rather than the integral in (7.6) is that the differential operators in (7.7) act on the eigenfunction ψ_λ rather than the call price $C(t,x;K)$. Derivatives of ψ_λ can be computed analytically, whereas derivatives of call prices $C(t,x;K)$ must be computed numerically from market data.

Using (7.6) one can obtain the value of $\phi(\lambda)$ for all $\lambda \in \sigma(-\mathcal{A}^1)$. This information is sufficient for constructing the FK transition density $p^{\phi,1}(t,x,y)$ and the transition density $p_S(t,x,y)$ of S. However, to compute the value of K_{var} we need the drift γ and the Lévy measure ν of the subordinator T. As we show in the next two subsections, γ and $\nu(ds)$ can be obtained from limited knowledge of the map ϕ.

7.1 Obtaining γ and ν from $\phi(\lambda)$: the compound Poisson case

As noted in section 2, when the subordinator T is of the compound Poisson type, its Lévy measure ν can be written as the product of the net jump intensity $\alpha := \nu((0,\infty))$ times the jump distribution F

$$\nu(ds) = \alpha\, F(ds). \tag{7.8}$$

In this scenario, the Lévy-Kintchine formula (2.10) can be written

$$\phi(\lambda) = \gamma\,\lambda + \alpha \int_0^\infty (1-e^{-\lambda s})F(ds) = \gamma\,\lambda + \alpha\left(1 - \widehat{F}(\lambda)\right), \tag{7.9}$$

where we have defined $\widehat{F}(\lambda)$, the Laplace transform of the measure $F(ds)$

$$\widehat{F}(\lambda) := \int_0^\infty e^{-\lambda s}F(ds).$$

The drift of the subordinator γ and the net jump intensity α can now be obtained from $\phi(\lambda)$ by taking the following limits

$$\lim_{\lambda\to\infty}\frac{\phi(\lambda)}{\lambda} = \lim_{\lambda\to\infty}\left(\gamma + \alpha\int_0^\infty \frac{1-e^{-\lambda s}}{\lambda}F(ds)\right) = \gamma,$$
$$\lim_{\lambda\to\infty}(\phi(\lambda) - \gamma\,\lambda) = \lim_{\lambda\to\infty}\alpha\int_0^\infty\left(1-e^{-\lambda s}\right)F(ds) = \alpha,$$

[5]Our thanks to Marco Avellaneda for pointing this out.

where we have used $\int_0^\infty F(ds) = 1$. After obtaining γ and α, we use (7.9) to solve for $\widehat{F}$

$$\widehat{F}(\lambda) = 1 + \frac{\gamma\lambda - \phi(\lambda)}{\alpha}. \tag{7.10}$$

Note, because $\sigma(-\mathcal{A}^1) \subseteq \mathbb{R}^+$, equation (7.10) will not give us the value of $\widehat{F}(\lambda)$ for any $\lambda < 0$ (we know $\widehat{F}(0) = 1$). However, knowledge of $\widehat{F}(\lambda)$ for $\lambda < 0$ is not needed in order to uniquely determine $\widehat{F}$. To see this, we need the following theorem.

Theorem 7.2 (Analyticity of the Laplace Transform)**.** *Let $\lambda \in \mathbb{R}$. Let $\alpha(s)$ be a real, non-negative, non-decreasing function which satisfies $\alpha(0) = 0$ and is of bounded variation on $[0, R]$ for every $R > 0$. If the integral*

$$\widehat{\alpha}(\lambda) := \int_0^\infty e^{-\lambda s}\alpha(ds),$$

converges for all $\lambda > \lambda_c$, $\lambda_c < \infty$ then $\widehat{\alpha}(\lambda)$ is analytic for all $\lambda > \lambda_c$, and

$$\frac{d^n}{d\lambda^n}\widehat{\alpha}(\lambda) := \int_0^\infty (-s)^n e^{-\lambda s}\alpha(ds). \tag{7.11}$$

Proof. See Widder (1946), page 57, Theorem 5a. □

Remark 7.3. A function that is analytic in a domain D is uniquely determined over D by its values along a line segment in D.

Remark 7.4. From (7.11), it is clear that $\widehat{\alpha}(\lambda)$ is decreasing and convex.

Let $\mathrm{Dom}(\widehat{F}) = (\lambda_c, \infty)$ where $\lambda_c \in \mathbb{R}$. If the continuous spectrum of $-\mathcal{A}^1$ is non-empty $\sigma_c(-\mathcal{A}^1) \neq \{\emptyset\}$, equation (7.10) gives us a map of $\widehat{F}(\lambda)$ on some interval $I \subset \mathrm{Dom}(\widehat{F})$. The analytic extension of that map is unique and well-defined in throughout $\mathrm{Dom}(\widehat{F})$. If the the continuous spectrum of $-\mathcal{A}^1$ is empty $\sigma_c(-\mathcal{A}^1) = \{\emptyset\}$, then (7.10) gives us a map of $\widehat{F}(\lambda)$ at a countably infinite number of points in $\mathrm{Dom}(\widehat{F})$ (i.e., the proper eigenvalues λ_n of $-\mathcal{A}^1$). In this case, the analytic extension of $\widehat{F}(\lambda)$ is still uniquely determined if F is Lipschitz (see Bäumer and Neubrander (1994), Corollary 1.3)

$$F(0) = 0 \qquad \text{and} \qquad \sup_{t,s\geq 0} \frac{|F(t) - F(s)|}{|t - s|} < \infty.$$

If F is not Lipschitz, the analytic continuation of $\widehat{F}(\lambda)$ is uniquely determined if we know the value of $\widehat{F}(\lambda)$ at equally spaced intervals, i.e., if, for $n = 1, 2, 3, \cdots$, we know $\widehat{F}(\lambda_n)$ where $\lambda_n = a + b\,n$ for some $a > \lambda_c$ and $b > 0$ (see Widder (1946), Theorem 6.2). Note that the eigenvalues (6.10) of the JDCEV process are equally spaced.

From a practical standpoint, one cannot evaluate (7.10) at an infinite number of λ. Thus, in order to obtain $F(ds)$ from (7.10), one should seek to fit a positive, analytic, decreasing, convex function to a finite number of points of $\widehat{F}(\lambda)$. Upon doing this, one can use the inverse Laplace transform (Bromwich integral) to obtain $F((0, s))$

$$F((0, s)) = \frac{1}{2\pi i}\int_0^s \left(\int_{C-i\infty}^{C+i\infty} e^{\lambda u}\widehat{F}(\lambda)d\lambda\right) du.$$

Here the constant $C \in \mathbb{R}$ is chosen so that the contour of integration lies to the right of all singularities of $\widehat{F}(\lambda)$. Another option for obtaining $F(ds)$ is to numerically invert the Laplace transform $\widehat{F}(\lambda)$. For a survey of numerical techniques for Laplace inversion we refer the reader to Davies and Martin (1979) and the references therein. Once $F(ds)$ is obtained, the Lévy measure ν is given by (7.8). In figure 1, we graphically illustrate how to obtain γ, α and $\widehat{F}(\lambda)$ from knowledge of $\phi(\lambda)$ at a discrete set of points.

7.2 Obtaining γ and ν from $\phi(\lambda)$: the general case

When the subordinator T is *not* of the compound Poisson type, its drift γ can still be found using

$$\lim_{\lambda\to\infty}\frac{\phi(\lambda)}{\lambda}=\lim_{\lambda\to\infty}\left(\gamma+\int_0^\infty\frac{1-e^{-\lambda s}}{\lambda}\nu(ds)\right)=\gamma.$$

To obtain ν we must introduce $\omega(s)$ the *tail of the Lévy measure*

$$\omega(s):=\nu((s,\infty))=\int_{(s,\infty)}\nu(dz).$$

Following Bertoin (2004), pp. 7, we note that

$$\frac{\phi(\lambda)}{\lambda}-\gamma=\int_{(0,\infty)}\frac{1-e^{-\lambda s}}{\lambda}\nu(ds)=\int_{(0,\infty)}e^{-\lambda s}\omega(s)ds=:\widehat{\omega}(\lambda), \tag{7.12}$$

where $\widehat{\omega}(\lambda)$ is the Laplace transform of the function $\omega(s)$. Depending on the nature of $\sigma(-\mathcal{A}^1)$, equation (7.12) either gives us a map of $\widehat{\omega}(\lambda)$ along a line segment $I\subset\mathrm{Dom}(\widehat{\omega})$, in which case the analytic extension of $\widehat{\omega}(\lambda)$ is unique, or (7.12) gives us the value of $\widehat{\omega}(\lambda)$ at a countably infinite number of points. In this case, the analytic continuation of $\widehat{\omega}(\lambda)$ is uniquely determined if we know the value of $\widehat{\omega}(\lambda)$ at equally spaced intervals (the Lipschitz condition would not be satisfied for an infinite activity Lévy process). From a practical standpoint, one may seek to fit an analytic, decreasing, convex function to a finite number of points of $\widehat{\omega}(\lambda)$. Upon doing this, one can obtain $\omega(s)$ from the Bromwich integral

$$\omega(s)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}e^{\lambda s}\widehat{\omega}(\lambda)d\lambda.$$

Finally, one obtains $\nu(ds)$ from

$$\nu(ds)=-d\omega(s).$$

8 Conclusion

In this paper we model the price process of an underlying S as a Feller diffusion time changed by a Lévy subordinator. This class of models, first developed in Mendoza-Arriaga et al. (2010), allows for the underlying to experience jumps with a state-dependent Lévy measure, local stochastic volatility and a local stochastic default intensity.

The contribution of this paper is two-fold. First, we show how to compute the price of a VS contract in the general Lévy subordinated diffusion setting. Using our general formula, we perform specific VS computations when the background diffusion is modeled as (i) a GBM with default and (*ii*) a JDCEV process. Second, we show that the drift and Lévy measure of the Lévy subordinator that drives the price process can be obtained dirctly by observation of the t-expiry volatility smile. By using call and put prices to uniquely determine the Lévy subordinator that drives the underlying price process, we reduce the risk of model misspecification.

Our results open the door for future theoretical and empirical research. On the theoretical side, we plan to extend our results to price options whose payoff depends on both the realized variance and the terminal value of the underlying. On the empirical side, we plan to study the extraction of the Lévy subordinator from market call and put data.

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A Proofs

A.1 Proof of Proposition 6.4

Part (i) is a straightforward computation

$$\int_0^t \mathbb{E}_x\left[\mathbb{I}_{\{\zeta^\phi>u\}}\gamma\sigma^2(X_u^\phi)\right]du = \gamma\sigma^2\int_0^t \mathbb{E}_x\left[\mathbb{I}_{\{\zeta^\phi>u\}}\right]du = \gamma\sigma^2\int_0^t e^{-ku}du = \gamma\sigma^2\left(\frac{1-e^{-kt}}{k}\right).$$

Part (ii). Using Proposition 4.3 we have

$$\begin{aligned}
&\int_0^t \mathbb{E}_x\left[\mathbb{I}_{\{\zeta^\phi>u\}}\int_{\mathbb{R}}\log^2\left(1+\frac{y}{X_{u-}^\phi}\right)\pi^{\phi,1}(X_{u-}^\phi,y)dy\right]du \\
&= \int_{(0,\infty)}\int_0^t\int_{\mathbb{R}} e^{-\lambda_\omega s}\Bigg\{\mathcal{P}_u^{\phi,1}\left(\psi_\omega(x)\right)\cdot\left(\int_{E\setminus\{x\}}\log^2(y)\overline{\psi}_\omega(y)\mathfrak{m}(y)dy\right) \\
&\quad - 2\,\mathcal{P}_u^{\phi,1}\left(\psi_\omega(x)\log x\right)\cdot\left(\int_{E\setminus\{x\}}\log(y)\overline{\psi}_\omega(y)\mathfrak{m}(y)dy\right) \\
&\quad + \mathcal{P}_u^{\phi,1}\left(\psi_\omega(x)\log^2 x\right)\cdot\left(\int_{E\setminus\{x\}}\overline{\psi}_\omega(y)\mathfrak{m}(y)dy\right)\Bigg\}d\omega\,du\,\nu(ds), \qquad \text{(A.1)}
\end{aligned}$$

where

$$\mathcal{P}_u^{\phi,1}\left(\psi_\omega(x)\log^n x\right) = (-i\sigma)^n\partial_\omega^n\left(e^{-\phi(\lambda_\omega)u}\psi_\omega(x)\right), \qquad \text{(A.2)}$$

$$\int_E \log^n(y)\overline{\psi}_\omega(y)\mathfrak{m}(y)dy = 2\sqrt{\pi}\,\sigma^{n-1/2}\,i^n\,\partial_\omega^n\delta(\omega+i\xi). \qquad \text{(A.3)}$$

Inserting (A.2) and (A.3) into (A.1) yields

$$\begin{aligned}
\text{(A.1)} &= \int_{(0,\infty)}\int_0^t\int_{\mathbb{R}} e^{-\lambda_\omega s}\Bigg\{\left(e^{-\phi(\lambda_\omega)u}\psi_\omega(x)\right)\cdot 2\sqrt{\pi}\,\sigma^{2-1/2}\,i^2\,\partial_\omega^2\delta(\omega+i\xi) \\
&\quad - 2\,(-i\sigma)\partial_\omega\left(e^{-\phi(\lambda_\omega)u}\psi_\omega(x)\right)\cdot 2\sqrt{\pi}\,\sigma^{1-1/2}\,i\,\partial_\omega\delta(\omega+i\xi) \\
&\quad + (-i\sigma)^2\partial_\omega^2\left(e^{-\phi(\lambda_\omega)u}\psi_\omega(x)\right)\cdot 2\sqrt{\pi}\,\sigma^{-1/2}\,\delta(\omega+i\xi)\Bigg\}d\omega\,du\,\nu(ds) \\
&= 2\sqrt{\pi}\sigma^{2-1/2}i^2\int_{(0,\infty)}\int_{\mathbb{R}}\Bigg\{\partial_\omega^2\left[e^{-\lambda_\omega s}\left(\frac{1-e^{-\phi(\lambda_\omega)t}}{\phi(\lambda_\omega)}\psi_\omega(x)\right)\right] \\
&\quad - 2\,\partial_\omega\left[e^{-\lambda_\omega s}\partial_\omega\left(\frac{1-e^{-\phi(\lambda_\omega)t}}{\phi(\lambda_\omega)}\psi_\omega(x)\right)\right] \\
&\quad + \left[e^{-\lambda_\omega s}\partial_\omega^2\left(\frac{1-e^{-\phi(\lambda_\omega)t}}{\phi(\lambda_\omega)}\psi_\omega(x)\right)\right]\Bigg\}\delta(\omega+i\xi)d\omega\,\nu(ds). \qquad \text{(A.4)}
\end{aligned}$$

In the second equality we have integrated with respect to u and used integration by parts to move the ∂_ω^n off of the Dirac delta functions. Noting that

$$(fg)'' - 2(fg')' + fg'' = f''g, \qquad \psi_{-i\xi}(x) = \sqrt{\sigma/4\pi}, \qquad \lambda_{-i\xi} = k,$$

we find

$$\begin{aligned}
\text{(A.4)} &= 2\sqrt{\pi}\sigma^{2-1/2}i^2\int_{(0,\infty)}\left\{\partial_\omega^2\left(e^{-\lambda_\omega s}\right)\cdot\left(\frac{1-e^{-\phi(\lambda_\omega)t}}{\phi(\lambda_\omega)}\psi_\omega(x)\right)\right\}_{\omega=-i\nu}\nu(ds) \\
&= \sigma^2\left(\frac{1-e^{-\phi(k)t}}{\phi(k)}\right)\int_{(0,\infty)} e^{-ks}\left(s+s^2\nu^2\right)\nu(ds).
\end{aligned}$$

A.2 Computation of $\tilde{c}_n = (x^p, \psi_n)$ from Proposition 6.7

Part (ii). Consider the series $U(x) = \sum_{k=1}^{\infty} u_k(x)$ for $x \in D \subset (0,\infty)$. If for all $x \in D$ the function $u_k(x)$ satisfies the inequality $|u_k(x)| \le d_k$ for $k = 1, 2, \cdots$, where the series $\sum_{k=1}^{\infty} d_k < \infty$, then the series $U(x)$ converges uniformly (see Prudnikov et al. (1990), Section I.3.4.3, p.751). From the inequality (27a) on p.54 of Nikiforov and Uvarov (1988), we find that $|\psi_n^1(x)| < C/(n-1)^{1/4}$ for some $C < \infty$ independent of n. Therefore, we have

$$|c_n \psi_n(x)| \le \frac{C\left| \left(\frac{1-p}{2|\beta|}\right)_{n-1} \right|}{\sqrt{(n-1)!\Gamma(\nu+n)}(n-1)^{1/4}} = d_n,$$

To show that the series $\sum_n^{\infty} d_n$ converges, it is enough to show that for a large n we have $\log(d_n)/\log(n) < -1$ (see Prudnikov et al. (1990) Section I.3.2.19, p.751). Therefore, observe that

$$d_n \approx \frac{(n-1)^{\frac{1-p}{2|\beta|}+n/2-3/2}}{(\nu+n)^{(\nu+n)/2-1/4}}, \qquad n \gg 1.$$

Thus

$$\begin{aligned}\frac{\log(d_n)}{\log(n)} &\approx \frac{\left(\left(\frac{1-p}{2|\beta|}\right) + n/2 - 3/2\right)\log(n-1) - ((\nu+n)/2 - 1/4)\log(\nu+n)}{\log(n)} \\ &< \Big[\left(\left(\frac{1-p}{2|\beta|}\right) + \frac{n}{2} - \frac{3}{2}\right) - \left(\frac{\nu+n}{2} - \frac{1}{4}\right) \Big] \frac{\log(n-1)}{\log(n)}.\end{aligned}$$

Note that $\log(n-1)/\log(n) \uparrow 1$. Moreover, it can be verified that the term inside the bracket is less than -1 for all $p > (\beta+1)/2 - c$ (recall $\nu = (2c+1)/(2|\beta|)$). This shows uniform convergence at $t = 0$ for $p > (\beta+1)/2 - c$. To show that the series is absolutely convergent at $t = 0$ it suffices to show that $\lim_{n\to\infty} d_{n+1}/d_n < 1$ with $d_n = |c_n \psi_n(x)|$ (i.e., d'Alambert's test for convergence). Equivalently, $\sum_n^{\infty} d_n$ converges if $\log(d_{n+1}/d_n) < 0$ as $n \to \infty$. Hence, analyzing the asymptotic behavior and noticing that $L_{n-1}^{\nu}(z) \approx e^{\frac{z}{2}} z^{-(2\nu+1)/4}(n-1)^{\nu/2-1/4} \cos\{2\sqrt{(n-1)z} - \pi(2\nu+1)/4\}/\sqrt{\pi}$ for $n \gg 1$, we find that

$$d_n = \Bigg| \frac{A^{\frac{\nu}{2} - \frac{p+2c}{2|\beta|}} \left(\frac{1-p}{2|\beta|}\right)_{n-1} \Gamma\left(\frac{p+2c}{2|\beta|} + 1\right)}{\sqrt{(n-1)!|\mu+b|\Gamma(\nu+n)}} \psi_n(x) \Bigg| < C\frac{(n-1)^{\frac{1-p}{2|\beta|}+\frac{\nu}{2}+n-\frac{7}{4}}}{(\nu+n)^{(\nu+n)-1/2}}, \qquad n \gg 1.$$

Thus, we would like to test $\lim_{n\to\infty} \log(d_{n+1}/d_n) < 0$, where $n \gg 1$. First observe that

$$\begin{aligned}\log\left(\frac{d_{n+1}}{d_n}\right) &= \left(\frac{1-p}{2|\beta|} + \frac{\nu}{2} + n - \frac{3}{4}\right)\log(n) - \left(\frac{1-p}{2|\beta|} + \frac{\nu}{2} + n - \frac{7}{4}\right)\log(n-1) \\ &\quad + \left(\nu + n - \frac{1}{2}\right)\log(\nu+n) - \left(\nu + n + \frac{1}{2}\right)\log(\nu+n+1).\end{aligned}$$

Then, making use of the approximation $\log(n+a) \approx \log(n) + a/n - a^2/(2n^2) + \cdots$, we find

$$\lim_{n\to\infty} \log\left(\frac{d_{n+1}}{d_n}\right) = \lim_{n\to\infty} -\frac{1}{n}\left(\frac{1-p}{2|\beta|} + \frac{\nu}{2} + n - \frac{7}{4}\right) + \left(\nu + n - \frac{1}{2}\right)\frac{\nu}{n} - \left(\nu + n + \frac{1}{2}\right)\frac{\nu+1}{n} = -2,$$

which concludes the proof.

A.3 Proof of Proposition 6.9

First observe that from Proposition 4.3 we obtain,

$$\begin{aligned}
&\int_0^t \mathbb{E}_x\left[\mathbb{I}_{\{\zeta^\phi>u\}}\int_{\mathbb{R}}\log^2\Big(1+\frac{y}{X^\phi_{u-}}\Big)\pi^{\phi,1}(X^\phi_{u-},y)dy\right]du\\
&=\int_{(0,\infty)}\int_0^t\sum_{n=1}^\infty e^{-\lambda_n s}\Big\{\Big(\int_{E\setminus\{x\}}\log^2(y)\psi_n(y)\mathfrak{m}(y)dy\Big)\big[\mathcal{P}^{\phi,1}_u\psi_n(x)\big]\\
&\quad-2\Big(\int_{E\setminus\{x\}}\log(y)\psi_n(y)\mathfrak{m}(y)dy\Big)\big[\mathcal{P}^{\phi,1}_u\big(\log(x)\psi_n(x)\big)\big]\\
&\quad+\Big(\int_{E\setminus\{x\}}\psi_n(y)\mathfrak{m}(y)dy\Big)\big[\mathcal{P}^{\phi,1}_u\big(\log^2(x)\psi_n(x)\big)\big]\Big\}du\,\nu(ds),
\end{aligned}\tag{A.5}$$

where

$$\begin{aligned}
\mathcal{P}^{\phi,1}_u\psi_n(x)&=\sum_{m=1}^\infty e^{-\phi(\lambda_m)u}(\psi_n,\psi_m)\psi_m(x)=e^{-\phi(\lambda_n)u}\psi_n(x),\\
\mathcal{P}^{\phi,1}_u\big(\log(x)\psi_n(x)\big)&=\sum_{m=1}^\infty e^{-\phi(\lambda_m)u}d^1_{n,m}\psi_m(x),\quad d^1_{n,m}=\int_E\log(y)\psi_n(y)\psi_m(y)\mathfrak{m}(y)dy,\\
\mathcal{P}^{\phi,1}_u\big(\log^2(x)\psi_n(x)\big)&=\sum_{m=1}^\infty e^{-\phi(\lambda_m)u}d^2_{n,m}\psi_m(x),\quad d^2_{n,m}=\int_E\log^2(y)\psi_n(y)\psi_m(y)\mathfrak{m}(y)dy.
\end{aligned}$$

Explicit representation of $d^1_{n,m}$ and $d^2_{n,m}$ are found using

$$d^1_{n,m}=2|\beta|A^{\nu+1}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E y^{2c-2\beta}\log(y)e^{-Ay^{-2\beta}}L^\nu_{n-1}(Ay^{-2\beta})L^\nu_{m-1}(Ay^{-2\beta})dy,$$

and

$$d^2_{n,m}=2|\beta|A^{\nu+1}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E y^{2c-2\beta}\log^2(y)e^{-Ay^{-2\beta}}L^\nu_{n-1}(Ay^{-2\beta})L^\nu_{m-1}(Ay^{-2\beta})dy.$$

Making the change of variable $z=Ay^{-2\beta}$ we obtain

$$\begin{aligned}
d^1_{n,m}&=\frac{1}{2|\beta|}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu(\log(z)-\log(A))e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz\\
&=\frac{1}{2|\beta|}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu\log(z)e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz\\
&\quad-\frac{\log(A)}{2|\beta|}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz,
\end{aligned}$$

and

$$
\begin{aligned}
d^2_{n,m} &= \frac{1}{(2|\beta|)^2}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu(\log(z)-\log(A))^2 e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz\\
&= \frac{1}{4|\beta|^2}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu \log^2(z)e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz\\
&\quad - \frac{\log(A)}{2|\beta|^2}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu \log(z)e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz\\
&\quad + \frac{\log^2(A)}{4|\beta|^2}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz.
\end{aligned}
$$

Now, we use the identity $\int_E z^\nu e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz = \delta_{n,m}\Gamma(\nu+n)/(n-1)!$ (where $\delta_{n,m}$ is the Kronecker delta) and the series expansion for the Generalized Laguerre polynomials ($L^\nu_n(z) = \Gamma(\nu+n+1)/n!\sum_{k=0}^n(-n)_k z^k/(\Gamma(\nu+k+1)k!)$) to obtain

$$
d^1_{n,m} = \frac{1}{2|\beta|}\left\{\sqrt{\frac{(m-1)!\Gamma(\nu+n)}{(n-1)!\Gamma(\nu+m)}}\sum_{k=0}^{n-1}\frac{(1-n)_k}{\Gamma(\nu+k+1)k!}\int_E z^{(\nu+k+1)-1}\log(z)e^{-z}L^\nu_{m-1}(z)dz - \log(A)\right\},
$$

and

$$
\begin{aligned}
d^2_{n,m} = \frac{1}{4|\beta|^2}\Bigg\{&\sqrt{\frac{(m-1)!\Gamma(\nu+n)}{(n-1)!\Gamma(\nu+m)}}\sum_{k=0}^{n-1}\frac{(1-n)_k}{\Gamma(\nu+k+1)k!}\int_E z^{(\nu+k+1)-1}\log^2(z)e^{-z}L^\nu_{m-1}(z)dz\\
&- 2\log(A)\sqrt{\frac{(m-1)!\Gamma(\nu+n)}{(n-1)!\Gamma(\nu+m)}}\sum_{k=0}^{n-1}\frac{(1-n)_k}{\Gamma(\nu+k+1)k!}\int_E z^{(\nu+k+1)-1}\log(z)e^{-z}L^\nu_{m-1}(z)dz + \log^2(A)\Bigg\}.
\end{aligned}
$$

Similarly, we obtain

$$
c_n = \int_E \psi_n(y)\mathfrak{m}(y)dy = \frac{A^{\frac{1-2c}{4|\beta|}}(1/(2|\beta|))_{n-1}\,\Gamma(c/|\beta|+1)}{\sqrt{(n-1)!|\mu+b|\Gamma(\nu+n)}},
$$

where c_n is found by setting $p=0$ in (6.11). Also,

$$
\begin{aligned}
\int_E \log(y)\psi_n(y)\mathfrak{m}(y)dy &= \frac{A^{\frac{1-2c}{4|\beta|}}}{2|\beta|}\sqrt{\frac{(n-1)!}{|\mu+b|\Gamma(\nu+n)}}\int_E z^{\frac{c+|\beta|}{|\beta|}-1}(\log(z)-\log(A))e^{-z}L^\nu_{n-1}(z)dz\\
&= \frac{A^{\frac{1-2c}{4|\beta|}}}{2|\beta|}\sqrt{\frac{(n-1)!}{|\mu+b|\Gamma(\nu+n)}}\int_E z^{\frac{c+|\beta|}{|\beta|}-1}\log(z)e^{-z}L^\nu_{n-1}(z)dz - \frac{\log(A)}{2|\beta|}c_n,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_E \log^2(y)\psi_n(y)\mathfrak{m}(y)dy &= \frac{A^{\frac{1-2c}{4|\beta|}}}{4|\beta|^2}\sqrt{\frac{(n-1)!}{|\mu+b|\Gamma(\nu+n)}}\int_E z^{\frac{c+|\beta|}{|\beta|}-1}(\log(z)-\log(A))^2e^{-z}L^\nu_{n-1}(z)dz\\
&= \frac{1}{4|\beta|^2}\Bigg\{A^{\frac{1-2c}{4|\beta|}}\sqrt{\frac{(n-1)!}{|\mu+b|\Gamma(\nu+n)}}\int_E z^{\frac{c+|\beta|}{|\beta|}-1}\log^2(z)e^{-z}L^\nu_{n-1}(z)dz\\
&\quad - 2\log(A)A^{\frac{1-2c}{4|\beta|}}\sqrt{\frac{(n-1)!}{|\mu+b|\Gamma(\nu+n)}}\int_E z^{\frac{c+|\beta|}{|\beta|}-1}\log(z)e^{-z}L^\nu_{n-1}(z)dz + \log^2(A)c_n\Bigg\}.
\end{aligned}
$$

Substituting the above expressions into Eq. (A.5) and using the relation (6.16) for $\Theta^1_{n,m}$, and $\Theta^2_{n,m}$ as well as equations (6.14) and (6.15), we arrive to the final expression (6.13).

B Spectral Theorem

In this appendix we summarize the theory of self-adjoint operators acting on a Hilbert space. A detailed exposition on this topic (including proofs) can be found in Reed and Simon (1980) and Rudin (1991).

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$. A *linear operator* is a pair $(\mathrm{Dom}(\mathcal{A}),\mathcal{A})$ where $\mathrm{Dom}(\mathcal{A})$ is a linear subset of $\mathcal{H}$ and $\mathcal{A}$ is a linear map $\mathcal{A}:\mathrm{Dom}(\mathcal{A})\to\mathcal{H}$. The *adjoint* of an operator $\mathcal{A}$ is an operator $\mathcal{A}^*$ such that $(\mathcal{A}f,g)=(f,\mathcal{A}^*g), \forall\, f\in\mathrm{Dom}(\mathcal{A}), g\in\mathrm{Dom}(\mathcal{A}^*)$, where

$$\mathrm{Dom}(\mathcal{A}^*) := \{g\in\mathcal{H} : \exists\, h\in\mathcal{H} \text{ such that } (\mathcal{A}f,g)=(f,h)\ \forall\, f\in\mathrm{Dom}(\mathcal{A})\}.$$

An operator $(\mathrm{Dom}(\mathcal{A}),\mathcal{A})$ is said to be *self-adjoint* in $\mathcal{H}$ if

$$\mathrm{Dom}(\mathcal{A})=\mathrm{Dom}(\mathcal{A}^*), \qquad (\mathcal{A}f,g)=(f,\mathcal{A}g) \qquad \forall\, f,g\in\mathrm{Dom}(\mathcal{A}).$$

Throughout this appendix, for any self-adjoint operator $\mathcal{A}$, we will assume that $\mathrm{Dom}(\mathcal{A})$ is a dense subset of $\mathcal{H}$. A densely defined self-adjoint operator is closed (see Rudin (1991), Theorem 13.9).

Given a linear operator $\mathcal{A}$, the *resolvent set* $\rho(\mathcal{A})$ is defined as the set of $\lambda\in\mathbb{C}$ such that the mapping $(\mathcal{A}-\mathrm{Id}\,\lambda)$ is one-to-one and $R_\lambda := (\mathcal{A}-\mathrm{Id}\,\lambda)^{-1}$ is continuous with $\mathrm{Dom}(R_\lambda)=\mathcal{H}$. The operator $R_\lambda:\mathcal{H}\to\mathcal{H}$ is called the *resolvent*. The *spectrum* $\sigma(\mathcal{A})$ of an operator $\mathcal{A}$ is defined as $\sigma(\mathcal{A}) := \mathbb{C}\setminus\rho(\mathcal{A})$. We say that $\lambda\in\sigma(\mathcal{A})$ is an *eigenvalue* of $\mathcal{A}$ if there exists $\psi\in\mathrm{Dom}(\mathcal{A})$ such that the *eigenvalue equation* is satisfied

$$\mathcal{A}\,\psi=\lambda\,\psi. \tag{B.1}$$

A function ψ that solves (B.1) is called an *eigenfunction* of $\mathcal{A}$ corresponding to λ. The *multiplicity* of an eigenvalue λ is the number of linearly independent eigenfunctions for which equation (B.1) is satisfied. The spectrum of an operator $\mathcal{A}$ can be decomposed into two disjoint sets called the *discrete* and *essential* [6] spectra: $\sigma(\mathcal{A})=\sigma_d(\mathcal{A})\cup\sigma_e(\mathcal{A})$. For a normal operator $\mathcal{A}$, a number $\lambda\in\mathbb{R}$ belongs to $\sigma_d(\mathcal{A})$ if and only if λ is an isolated point of $\sigma(\mathcal{A})$ and λ is an eigenvalue of finite multiplicity (see Rudin (1991), Theorem 12.29).

A *projection-valued measure* on the measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is a family of bounded linear operators $\{E(B), B\in\mathcal{B}(\mathbb{R})\}$ in $\mathcal{H}$ that satisfies:

1. $E(\emptyset)=0$ and $E(\mathbb{R})=\mathrm{Id}$.
2. $E(B)$ is an orthogonal projection. That is, $E^2(B)=E(B)$ and $E(B)$ is self-adjoint: $E^*(B)=E(B)$.
3. $E(A\cap B)=E(A)E(B)$.
4. If $B=\bigcup_{i=1}^\infty B_i$ and $B_i\cap B_j=\emptyset$ for $i\neq j$ then $E(B)=\lim_{n\to\infty}\sum_{i=1}^n E(B_j)$, where the limit is in the strong operator topology.
5. For every in $f,g\in\mathcal{H}$ the set function $\mu_{f,g}(B) := (f,E(B)g)$ is a complex measure on $\mathcal{B}(\mathbb{R})$.

Theorem B.1 (Spectral Representation Theorem)**.** *There is a one-to-one correspondence between self-adjoint operators $\mathcal{A}$ and projection-valued measures E on $\mathcal{H}$, the correspondence being given by*

$$\mathcal{A}=\int_{\sigma(\mathcal{A})}\lambda\,E(d\lambda).$$

If $g(\cdot)$ is a Borel function on $\mathbb{R}$ then

$$g(\mathcal{A})=\int_{\sigma(\mathcal{A})} g(\lambda)E(d\lambda), \qquad \mathrm{Dom}(g(\mathcal{A}))=\{f\in\mathcal{H} : \int_{\sigma(\mathcal{A})}|g(\lambda)|^2\mu_{f,f}(d\lambda)<\infty\}. \tag{B.2}$$

[6] The essential spectrum may be further decomposed into the *continuous* spectrum and the *residual* spectrum. It can be shown that the residual spectrum of an ordinary differential operator is empty (see Roach (1982), page 184).

Proof. See Rudin (1991) Theorems 12.21 and 13.33. $\square$

As a practical matter, if $\mathcal{A}$ is a differential operator acting on a Hilbert space $L^2(I, \mathfrak{m}(x)dx)$, where I is an interval with endpoints $l < r$, then the operators defined by (B.2) can be constructed by solving the *proper* and *improper* [7] eigenvalue problems

$$\begin{aligned}
&\text{proper:} && \mathcal{A}\,\psi_n = \lambda_n\,\psi_n, && \lambda_n \in \sigma_d(\mathcal{A}), && \psi_n \in \mathcal{H}, \\
&\text{improper:} && \mathcal{A}\,\psi_\omega = \lambda_\omega\,\psi_\omega, && \lambda_\omega \in \sigma_e(\mathcal{A}), && \psi_\omega \notin \mathcal{H}.
\end{aligned}$$

For the improper eigenvalue problem one extends the domain of $\mathcal{A}$ to include functions all functions f for which the following boundedness conditions are satisfied

$$\lim_{x \searrow l} |f(x)|^2 \mathfrak{m}(x) < \infty, \qquad \lim_{x \nearrow r} |f(x)|^2 \mathfrak{m}(x) < \infty, \tag{B.3}$$

We will use Latin subscripts (e.g., l, m, n) to denote proper eigenfunctions/values and Greek subscripts (e.g., ω, ν, μ) to denote improper eigenfunctions/values. When we do not wish to distinguish between proper and improper eigenfunctions/values we will write ψ_λ and λ with no subscript. We refer to ψ_λ and λ as *generalized* eigenfunctions/values.

After normalizing, the proper and improper eigenfunctions $\mathcal{A}$ satisfy the following orthogonality relations

$$(\psi_n, \psi_m) = \delta_{n,m}, \qquad (\psi_\omega, \psi_\mu) = \delta(\omega - \mu), \qquad (\psi_n, \psi_\omega) = 0.$$

The operator $g(\mathcal{A})$ in (B.2) is constructed as follows (see Hanson and Yakovlev (2002), section 5.3.2)

$$\begin{aligned}
g(\mathcal{A})f &= \sum_\lambda g(\lambda)(\psi_\lambda, f)\psi_\lambda \\
&:= \sum_{\lambda_n \in \sigma_d(\mathcal{A})} g(\lambda_n)\,(\psi_n, f)\,\psi_n + \int_{\lambda_\omega \in \sigma_e(\mathcal{A})} g(\lambda_\omega)\,(\psi_\omega, f)\,\psi_\omega d\omega.
\end{aligned}$$

It is not always easy to evaluate divergent integrals of the form $(\psi_\lambda, \psi_{\lambda'})$ and verify that they are in fact delta functions $\delta(\lambda - \lambda')$. A method for directly obtaining properly normalised improper eigenfunctions can be found on page 238 of Friedman (1956).

[7] The term "improper" is used because the improper eigenvalues $\lambda \notin \sigma_d(\mathcal{A})$ and the improper eigenfunctions $\psi_\lambda \notin \mathcal{H}$ since $(\psi_\omega, \psi_\omega) = \infty$.

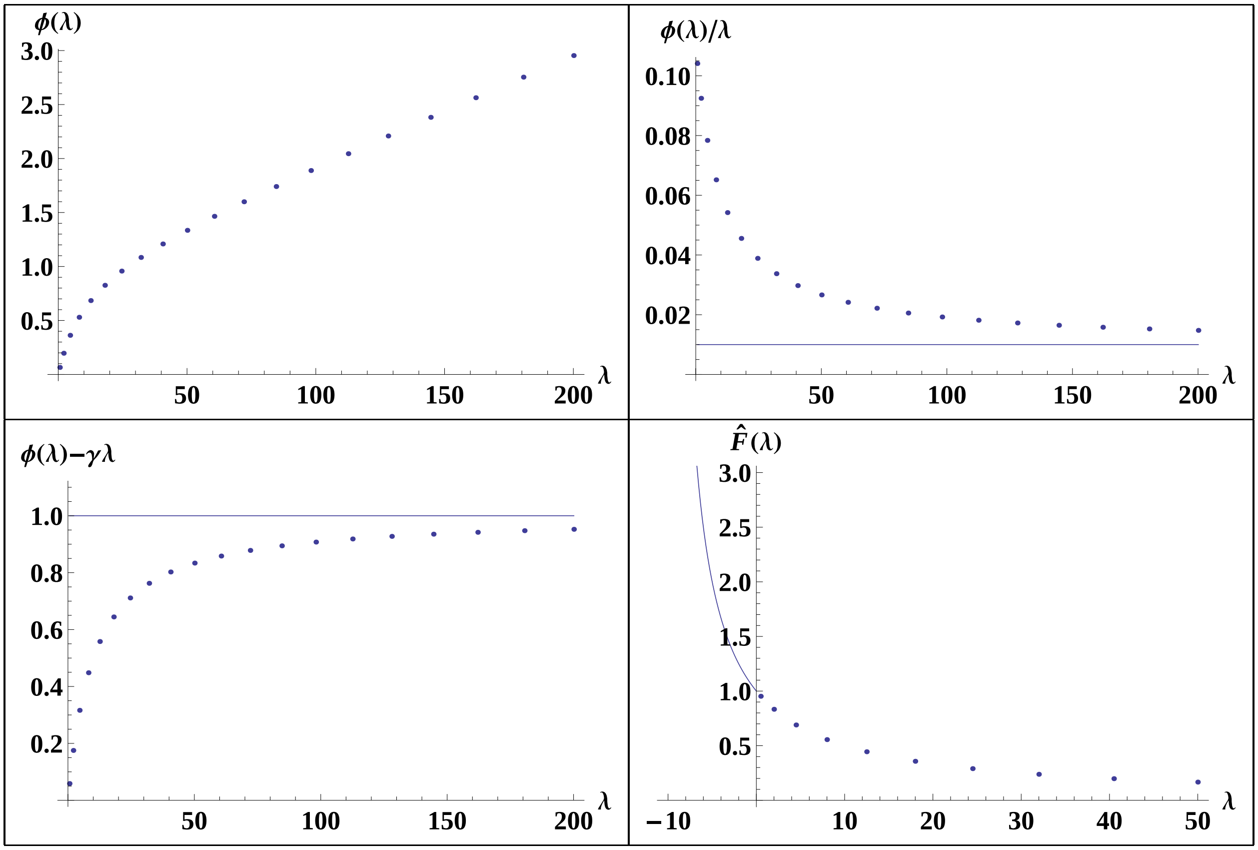

Figure 1: A graphical illustration of how to obtain the drift γ, the jump intensity α and the Laplace transform $\widehat{F}(\lambda)$ of the jump distribution $F(ds)$ for a subordinator of the compound Poisson type from a discrete set of values of the subordinator's Laplace exponent $\phi(\lambda)$. Top left: a plot of $\phi(\lambda)$ for a Lévy subordinator with exponentially distributed jumps $\nu(ds) = \alpha\eta e^{-\eta s} ds$. Top right: a plot of $\phi(\lambda)/\lambda$ for the same subordinator. The level of the solid line corresponds to drift of the subordinator γ. Bottom Left: a plot of $\phi(\lambda) - \gamma\lambda$ for the same subordinator. The level of the solid line corresponds to the net jump intensity α. Bottom right: a plot of $\widehat{F}(\lambda)$ and its analytic continuation for values of $\lambda < 0$. In all four plots we use parameter values $\gamma = 1/100$, $\alpha = 1$ and $\eta = 10$.

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